

General analysis of type I second-harmonic generation with elliptical Gaussian beams

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One can enhance second-harmonic generation in angle-tuned crystals by focusing the fundamental to an elliptical waist in the crystal. We present a general analysis for type I phase-matched uniaxial and biaxial crystals that will find particular application in the generation of harmonics from diode lasers. © 1997 Optical Society of America [S0740-3224(97)01308-8]

1. INTRODUCTION

The optimal focusing arrangement for second-harmonic generation is a compromise between the focal intensity and the depth of focus (confocal parameter) within the crystal. In the absence of birefringence (noncritical phasematching), the fundamental beam should have a circular cross section and a confocal parameter of 0.35 of the crystal's length.¹

Angle-tuned (critical) phase matching causes the cylindrical symmetry of harmonic generation to be lost, and elliptical (or anamorphic) focusing into the crystal has been shown to be advantageous. In this paper we present a theoretical analysis of type I phase matching and conclude that the harmonic power can be enhanced by as much as 30%. The less common type II phase matching, in which the fundamental is split between ordinary and extraordinary polarizations, requires a rather different analysis if these components are collinear²; a proposed noncollinear scheme is, however, equivalent to the type I arrangement considered here.³

Elliptical focusing will be especially useful for the frequency doubling of diode lasers,⁴ for not only is the beam profile usually elliptical but any enhancement cavities are likely to be astigmatic. To match such a geometry efficiently, or at least to be allowed a degree of insensitivity, will be useful indeed. Such considerations will be yet more important when frequency-doubling or -mixing stages are cascaded, as with frequency tripling⁵ and quadrupling.^{6,7} Our results will also be of particular relevance to second-harmonic generation by use of Brewster-angled crystals that render a circular incident beam elliptical. For high-power lasers as well, if the optimal circular focus leads to crystal damage⁸ or to thermally induced dephasing, elliptical focusing may provide a solution.

Cylindrical focusing was first considered for second-harmonic generation by Volosov and Nilov,⁹ Volosov,¹⁰

and Chmela,¹¹ but true elliptical arrangements were not addressed until Librecht and Simons¹² considered a specific example and analyzed the asymptotic behavior. Elliptical geometries had in the meantime been applied to parametric oscillation by Kuizenga,¹³ and further analyses followed.^{14,15} Pure cylindrical focusing has been used with pulsed Nd:YAG lasers¹⁶ and with copper-vapor lasers,¹⁷ for which a limited theoretical analysis has been presented¹⁸ and proposed for the reduction of thermal dephasing.¹⁹ Interest in elliptical focusing has reawakened with the advent of the diode laser, first with the investigation of a specific example of type II phase matching²⁰ and then with the qualitative matching of an elliptical beam into an enhancement cavity²¹; the latter technique was also partly addressed in conjunction with titanium-doped sapphire lasers.^{22,23} Recently Steinbach *et al.*²⁴ showed excellent agreement between modeled and experimental doubling of a cw argon-ion laser and calculated some more general results.

This is clearly, then, not the first attempt to treat elliptical focusing in second-harmonic generation. Whereas previous analyses generally referred to specific experimental situations, however, the treatment presented here leads to results of general significance as a function of the usual crystal parameter B , from which the optimal elliptically focused geometries can readily be found and for which generally applicable recipes are presented. Often, of course, it may not be experimentally practicable to use the ideal arrangement, and the conversion efficiency indeed appears to be a gentle function of the focusing parameters. We therefore present curves for nonoptimal ellipticities from which the best focusing strengths and expected efficiencies can be derived.

Finally, in various limiting regimes, some physical interpretation of the results is possible. For many nonlinear crystals the birefringence is high enough that an analytic calculation for the limit of high birefringence is a

good approximation. The treatment given here, which reduces the calculation to tabulated functions, follows that of Librecht and Simons,¹² although it is appropriate not only in the extremes of strong and weak focusing but also in the interesting regime where the conversion efficiency is maximum.

2. THEORY

Second-harmonic generation is most efficient when it is phase matched; that is, the material dispersion between the fundamental and the harmonic wavelengths is compensated such that, when the harmonic contributions from all points within the crystal are added coherently into the output beam, the maximum intensity is achieved. For plane waves this requires that the components all emerge in phase and thus that the refractive indices for the fundamental and the harmonic be equal. For focused beams there should be a small mismatch to correct for the Guoy shift,²⁵ which can be considered to conserve the momentum of the light beam as it passes through a focus.

Type I phase matching exploits the birefringence of the nonlinear crystal, which is oriented such that the refractive index for one polarization at the fundamental wavelength matches that at the harmonic in the other. If the refractive indices for the two polarizations are represented as functions of the angular coordinates by surfaces, then phase matching occurs when the ellipsoids thus formed meet. If these surfaces intersect, the line of intersection corresponds to the locus of phase-matched crystal orientations, and rotation normal to this line allows the crystal to be angle tuned, the orientation chosen being that point on the locus that provides the largest nonlinear coefficient. Except in the case of noncritical, or 90°, phase matching—which involves bringing the ellipsoids to a single grazing contact by varying the crystal temperature—the phase match will be highly sensitive to rotation away from the line of intersection of the ellipsoids, causing a narrow angular acceptance for the incident light, but will be insensitive to rotation along this line.

The induced field at the harmonic frequency depends quadratically on that of the fundamental, and the efficiency of second-harmonic generation thus increases as the input power is focused into a smaller area. If the constraints of phase matching can be neglected, the optimal focusing arrangement will be a compromise that optimizes both the electric-field strength at the focus and the longitudinal distance over which the focus extends and occurs when the confocal parameter of the Gaussian beam is 0.35 of the crystal length. With critical phase matching, however, only a central longitudinal slice of the beam falls within the narrow acceptance angle, and for optimal conversion the focusing must be weakened. The correspondingly larger focal region can be regarded as better able to accommodate Poynting-vector walk-off between the beams. The optimal parameters for a circular Gaussian beam in any such arrangement are described in a classic paper by Boyd and Kleinman.¹

3. ANALYSIS

Our approach has been to generalize the circular beam analysis of Boyd and Kleinman,¹ using instead an ellipti-

cal, astigmatic form of the Gaussian beam to describe the fundamental and generated harmonics. We preserve the original notation of Boyd and Kleinman but have chosen to present our results in SI units. In extending the treatment to allow the Gaussian beams to have an elliptical cross section, we have added x and y subscripts where the elliptical asymmetry renders it necessary. The optic axis is in the x - z plane.

We start, therefore, with an elliptical Gaussian beam, propagating with wave number $k_1 = 2\pi\eta_1/\lambda$ along the z' axis, with an electric field given by

$$E = E_0 \frac{\exp(ik_1z') \exp\left[-\left(\frac{x'^2/w_x^2}{1+i\tau_x'} + \frac{y'^2/w_y^2}{1+i\tau_y'}\right)\right]}{[(1+i\tau_x')(1+i\tau_y')]^{1/2}}, \quad (1)$$

where $\tau_x' = 2(z' - f_x)/k_1w_x^2$ and $\tau_y' = 2(z' - f_y)/k_1w_y^2$. The foci in the y - z and x - z planes are found at $z = f_x$ and $z = f_y$, respectively, thus permitting the beam to be astigmatic when the foci do not coincide, and the waist radii or ellipse semiaxes are w_x and w_y , respectively.

Birefringence in the nonlinear crystal will be characterized by the walk-off angle ρ between the fundamental and the harmonic. For type I phase matching in negative uniaxial crystals, where the ordinary (o) and the extraordinary (e) refractive indices at the fundamental and the harmonic frequencies are given by η_o^ω , η_e^ω , $\eta_o^{2\omega}$ and $\eta_e^{2\omega}$, the walk-off angle is given by

$$\tan \rho = \frac{1 - (\eta_o^{2\omega}/\eta_e^{2\omega})^2}{\cot \theta + \tan \theta (\eta_o^{2\omega}/\eta_e^{2\omega})^2}, \quad (2)$$

where the phase-matching angle θ satisfies

$$\sin^2 \theta = \frac{(\eta_o^\omega)^{-2} - (\eta_o^{2\omega})^{-2}}{(\eta_e^{2\omega})^{-2} - (\eta_o^{2\omega})^{-2}}. \quad (3)$$

Similar expressions apply for positive uniaxial crystals, with the ω and 2ω labels interchanged, and the treatment presented here is valid for both. With biaxial crystals^{26,27} there are rather too many configurations to describe here, but one can derive a value for the walk-off angle ρ by using

$$\begin{aligned} \tan \rho &= \frac{1}{k_2} \frac{d(\Delta k)}{d\phi} \\ &= \frac{1}{\eta} \frac{d}{d\phi} [\eta^\omega(\phi) - \eta^{2\omega}(\phi)], \end{aligned} \quad (4)$$

where ϕ is the angular coordinate in the critical direction, $\eta^\omega(\phi)$ and $\eta^{2\omega}(\phi)$ are the angle-dependent refractive indices at the fundamental and the harmonic frequencies, respectively, and Δk is the wave-number mismatch, $2k_1 - k_2$.

We now apply the analysis of Boyd and Kleinman, using the more general elliptical form of the Gaussian beam. The fundamental electric field at a given distance into the crystal is given by Eq. (1) together with a term to allow for absorption, and the resulting polarization at the harmonic frequency is found to have a transverse elliptical

Gaussian dependence. Each slice will thus propagate as an elliptical Gaussian beam, and all such beams can be added coherently further downstream; walk-off is included by making a shear coordinate transformation. Our version of the variables used in this treatment is summarized as follows:

$\omega_1, \omega_2,$	fundamental and harmonic frequencies;
$k_1, k_2,$	fundamental and harmonic wave numbers;
$\Delta k,$	wave-number mismatch, $2k_1 - k_2$;
$w_x, w_y,$	waist radii (ellipse semi-axes);
$b_x, b_y,$	confocal parameters, $w_x^2 k_1, w_y^2 k_1$;
$l,$	crystal length;
$\xi_x, \xi_y,$	focusing parameters, $l/b_x, l/b_y$;
$\mathbf{e},$	ellipticity, w_x/w_y ;
$\mu,$	focus position in the critical (x) direction = $(l - 2f_x)/l$ when the focus is at $z = f_x$;
$\Delta,$	astigmatic distance between beam waists = $2(f_x - f_y)/(k_1 w_y^2)$;
$\rho,$	birefringent walk-off angle in the crystal;
$\beta,$	ratio of walk-off angle to Gaussian-beam half-angle, $=\rho w_x k_1/2$.

\mathbf{e} is thus the ellipticity of the Gaussian beam waist. Two dimensionless coordinate systems are used: s and s' are angles from the axis in the critical and noncritical planes, in units of Gaussian half-angle; $\tau_x, \tau_x', \tau_y,$ and τ_y' are longitudinal dimensions in units of the confocal parameter. The remaining variables are essentially unchanged:

$\sigma_x,$	phase mismatch in the critical direction = $1/2 b_x \Delta k$;
σ'	= $\sigma_x + 4\beta s$;
K	= $2\omega_1^2 d_{\text{eff}}^2 / \pi \epsilon_0 c^3 \eta_1^2 \eta_2$;
$P_1,$	fundamental power;
$\alpha_1,$	absorption coefficient for fundamental;
$\alpha_2,$	absorption coefficient for harmonic;
α	= $\alpha_1 - 1/2 \alpha_2$;
α'	= $\alpha_1 + 1/2 \alpha_2$;
κ	= $1/2 \alpha b_x$.

The relations among the on-axis focal harmonic polarization P_{0x} , the fundamental electric field E_0 , and the total fundamental power P_1 are given by

$$P_{0x} = \epsilon_0 d_{\text{eff}} E_0^2 = 4P_1 d_{\text{eff}} / (\pi \eta_1 c w_x w_y), \quad (6)$$

and the second harmonic intensity S is given in terms of the field strength E_2 by

$$S = 1/2 \epsilon_0 \eta_2 c |E_2|^2. \quad (7)$$

We ultimately reach an expression for the harmonic power:

$$P_2 = KP_1^2 l k_1 \exp(-\alpha' l) h_m, \quad (8)$$

where all the optimizable parameters are contained within the term

$$h_m = \frac{\pi^2}{\xi_x} \exp(\mu \alpha l) \left[\frac{2}{\pi^{1/2}} \mathbf{e} \int_{-\infty}^{\infty} \exp(-4s^2) |H|^2 ds \right], \quad (9)$$

where

$$H = \frac{1}{2\pi} \int_{-\xi_x(1-\mu)}^{\xi_x(1+\mu)} \frac{\exp(-\kappa \tau_x') \exp(i\sigma' \tau_x')}{(1 + i\tau_x')^{1/2} [1 + i(e^2 \tau_x' + \Delta)]^{1/2}} d\tau_x'. \quad (10)$$

Although we have chosen throughout to refer primarily to properties in the critical x direction, a similar pair of expressions results if we work instead with properties in the y direction and make appropriate changes to our definitions of $\beta, \kappa, \mu, \sigma',$ and \mathbf{e} .

h_m is a function of the parameters $\xi_x, \mathbf{e}, \beta, \sigma_x, \mu, \Delta, l,$ and α . The last term characterizes the absorption of the crystal and may often be assumed to be zero, in which case the optimal values of μ and Δ are found to be zero and the dependence of h_m on l will disappear. σ_x represents the deviation from normal phase matching and will in practice be optimized by fine adjustment of the crystal orientation, so only the optimal value of σ_x need be considered. The set of parameters necessary for determining h_m is thus reduced to $\xi_x, \mathbf{e},$ and $\beta,$ and it is convenient to rewrite β in the form

$$\beta = B(\xi_x)^{-1/2}, \quad (11)$$

where B is completely defined by the physical properties of a given crystal through $B = 1/2 \rho(lk_1)^{1/2}$. B is thus the ratio of the walk-off angle to the $1/e$ amplitude half-angle of a Gaussian beam whose confocal parameter equals the crystal length; alternatively, B is $\sqrt{1.39}$ times this ratio for a beam that would be the optimum under noncritical phase matching. The function h_m is thus defined by $B, \xi_x,$ and $\mathbf{e},$ where the first parameter describes the crystal and the others describe the focusing arrangement, which is under our control.

Equation (8) correctly reduces to the result of Boyd and Kleinman for circular focusing and is consistent with those of Steinbach *et al.*²⁴ when there is no absorption and of Librecht and Simons¹² when in addition the focus is at the crystal center. The formula of Bourzeix *et al.*²³ describes the behavior in the limit $\mathbf{e} \rightarrow 0$ and provides a reasonable estimate of the efficiency for $\mathbf{e} < 1$ but is inappropriate at the optimal ellipticity and above and cannot be used to derive the optimal conditions.

For the numerical calculation we first expand Eqs. (9) and (10) as a triple integral over s and τ_x' (twice, as τ and τ'), which allows the s integral to be eliminated and renders the remaining double integral better behaved. By making the transformations $X = \tau - \tau'$ and $Y = \tau + \tau' - 2\xi_x \mu$, we move the range in which the integrated function is large onto the $X = 0$ axis. We then compute h_m numerically, using the amoeba downhill simplex method²⁸ to maximize the result for specific B as a function of $\xi_x, \sigma_x,$ and \mathbf{e} . For the results presented, we have set the absorption to zero and placed the foci at the center of the crystal, which makes the calculation slightly easier and gives the results some general significance.

4. RESULTS AND INTERPRETATION

The maximized values of h_m for circular and optimal elliptical focusing are shown in Fig. 1 as functions of the crystal parameter B , and the corresponding focusing parameters, ξ for circular and ξ_x and ξ_y for elliptical focusing, are shown in Fig. 2. The optimal ellipticity is indicated by the heavy solid curves of Figs. 3 and 4, which show the values of h_m and, respectively, ξ_y and ξ_x as functions of both B and e and permit the performance of geometries with nonoptimal ellipticity to be found. The observed improvement in conversion efficiency, approaching a factor of 1.3,^{21,24,29} is indeed predicted for elliptical focusing in crystals with appreciable birefringence and corresponds as expected to stronger focusing in the noncritical direction and to a broader focus in the critical direction. In such cases the conversion efficiency is little affected by a further increase in ellipticity, and it is found that w_y remains fixed near its optimal value and thus that $\xi_y = e^2 \xi_x$ remains constant, as described by Steinbach *et al.*²⁴; for suboptimal ellipticities, however, ξ_y should decrease to meet the results of Boyd and Kleinman at $e = 1$.

Our results can be compared with those for parametric gain,¹³ which corresponds to generation of the harmonic in a particular Gaussian mode. Our cases of circular and elliptical focusing are given, respectively, by Kuizenga's configurations¹³ for circular and elliptical probe beams with elliptical signal and idler, and α_{12m} corresponds to

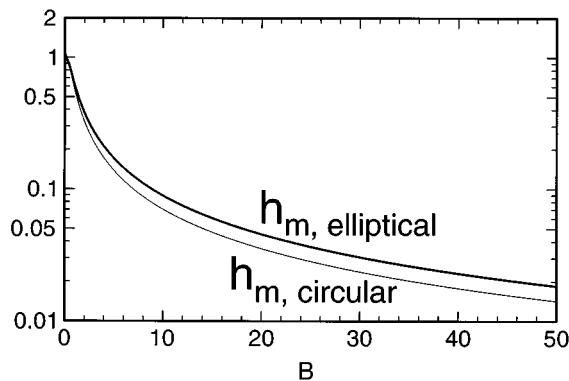


Fig. 1. Optimized conversion efficiency parameter h_m for circular and elliptical Gaussian beams as a function of the crystal parameter B .

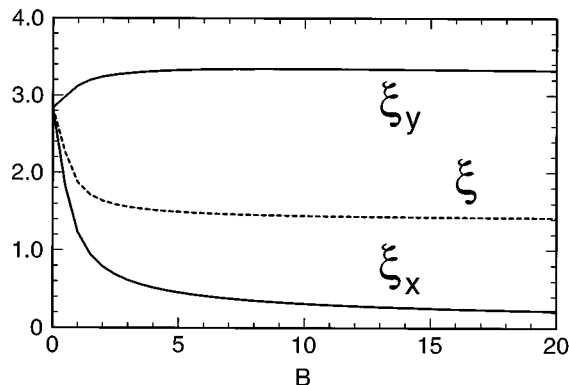


Fig. 2. Optimal focusing parameters ξ_x and ξ_y for elliptical Gaussian beams and ξ for circular focusing.

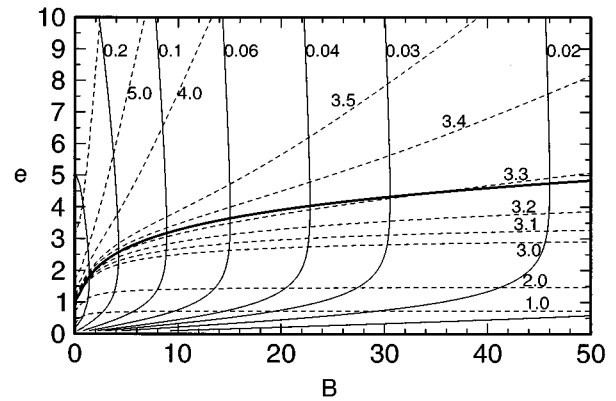


Fig. 3. Contours of h_m (solid curves) and the focusing parameter ξ_y (dashed curves) as functions of ellipticity e and crystal parameter B for optimized second-harmonic generation with elliptical Gaussian beams. The heavy solid curve shows the optimal ellipticity as a function of B . Contour values are 0.02, 0.03, 0.04, 0.06, 0.1, 0.2 and 0.5 for h_m and 1.0, 2.0, 3.0, 3.1, 3.2, 3.3, 3.4, 3.5, 4.0, 5.0 and 10.0 for ξ_y .

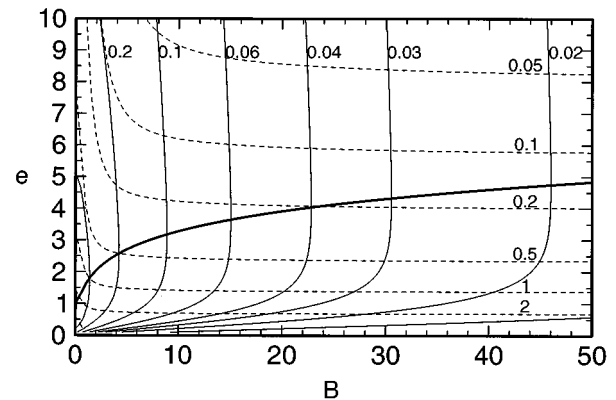


Fig. 4. Contours of h_m (solid curves) and the focusing parameter ξ_x (dashed curves) as functions of ellipticity e and crystal parameter B for optimized second-harmonic generation with elliptical Gaussian beams. The heavy solid curve shows the optimal ellipticity as a function of B . Contour values are 0.02, 0.03, 0.04, 0.06, 0.1, 0.2 and 0.5 for h_m and 0.05, 0.1, 0.2, 0.5, 1.0, 2.0 and 5.0 for ξ_x .

the reciprocal of our ellipticity e . Kuizenga's results, which closely follow ours, also indicate the ellipticity of the harmonic beam $1/(\alpha_{12}\sqrt{\alpha_x})$. At high B , the parametric gain calculations indicate an optimal efficiency approximately 10% lower than ours, consistent with an experimentally determined coupling efficiency of 87%.⁷

Boyd and Kleinman¹ derive asymptotic forms for the dependence of h_m on ξ_x for the cases of weak and strong birefringence and focusing. Without birefringent walk-off (noncritical phase matching), weak focusing corresponds to the case of bounded plane waves when the harmonic power varies as the square of the crystal length and hence, for given $w_{x,y}$, in proportion to ξ_x ; with strong focusing, the focal region lies completely within the crystal, and the independence of the harmonic power from the crystal length requires that h_m vary in inverse proportion to ξ_x . With strong birefringence, walk-off limits the range of coherent addition to a length $l_a = \pi^{1/2} w_x / \rho$, and the asymptotic dependences thus become $\sqrt{\xi_x}$ and $1/\sqrt{\xi_x}$.

Such an analysis can be applied to elliptical Gaussian beams. For the case of high B and $\mathbf{e} \gg 1$, we assume the harmonic intensity to correspond to the incoherent sum of coherent parcels of length l_a within an overall length b_y (the beam is more strongly focused in the noncritical y direction), and we take the fundamental intensity to be constant in this region and confined to an area with the waist dimensions. The harmonic is assumed to fill an angular divergence matching that of the fundamental. Combining these terms gives a second-harmonic power of

$$P_2 \propto \left[\left(\frac{P_1}{w_x w_y} \right) w_x w_y l_a \left(\frac{b_y}{l_a} \right)^{1/2} \right]^2 \left(\frac{1}{w_x w_y} \right), \quad (12)$$

$$\propto P_1^2 w_y / \rho. \quad (13)$$

Effectively, we have decoupled the x and y dependences, so w_y takes its optimal value and the efficiency is independent of w_x . Inasmuch as $P_2 \propto l h_m$,

$$h_m \propto 1/B \sqrt{\xi_y}. \quad (14)$$

This behavior is apparent in the upper-right-hand quadrant of Fig. 3, where ξ_y is essentially constant and h_m is independent of ellipticity. The analysis is, however, valid only when $l_a \ll b_y$, corresponding to the condition that $\mathbf{e} \ll B \sqrt{\xi_y}$, which accounts for the existence of an optimal ellipticity. Beyond this limit, the beam is cylindrically focused and the efficiency varies in inverse proportion to the ellipticity.

The derivation of relation (14) corresponds to the Boyd–Kleinman case of high β , whereby with increasing birefringence it becomes appropriate to consider only the part of the Gaussian whose direction lies within the acceptance angle of the crystal. In this regime, Boyd and Kleinman find that an analytic solution is possible; the same is true in the generalized case with elliptical Gaussian beams.

5. ANALYTIC SOLUTION FOR HIGH β

Once again ignoring absorption, we position the foci at the center of the crystal and consider the regime where β is sufficiently large that $\exp(-4s^2)$ ($=\exp\{-4 \times [(\sigma' - \sigma_x)/4\beta]^2\}$) is essentially unity over the range of significant $|H|^2$. This means that within the crystal's acceptance angle the beam has uniform amplitude; equivalently, the length for coherent addition l_a should be less than the crystal length l . The condition that $l_a \ll b_y$ again applies, and we assume strong focusing ($b_x, b_y \ll l$) but continue to make the paraxial approximation that has been assumed throughout this treatment: modifications for the case of a large walk-off angle have been considered by Eckardt *et al.*³⁰ h_m can now be written as

$$h_m = \frac{2\pi^{3/2}\mathbf{e}}{\xi_x} \int_{-\infty}^{\infty} \exp\left[-4\left(\frac{\sigma' - \sigma}{4\beta}\right)^2\right] \times \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i\sigma' \tau_x' f(\tau_x')] d\tau_x' \right|^2 \frac{d\sigma'}{4\beta}, \quad (15)$$

with

$$f(\tau_x') = \frac{1}{(1 + i\tau_x')^{1/2}(1 + i\mathbf{e}^2\tau_x')^{1/2}} g(\tau_x'), \quad (16)$$

where the aperture function $g(\tau_x') = 1$ for $|\tau_x'| < \xi_x$ and $g(\tau_x') = 0$ elsewhere. We now invoke Parseval's theorem and find that

$$h_m = \frac{\pi^{1/2}\mathbf{e}}{4\beta\xi_x} \int_{-\infty}^{\infty} |f(\tau_x')|^2 d\tau_x' = \frac{\pi^{1/2}\mathbf{e}}{4\beta\xi_x} \int_{-\xi_x}^{\xi_x} \left| \frac{1}{(1 + i\tau_x')^{1/2}(1 + i\mathbf{e}^2\tau_x')^{1/2}} \right|^2 d\tau_x'. \quad (17)$$

The function of τ can be rewritten as

$$\left| \left[\frac{1}{(1 + i\tau_x')(1 + i\mathbf{e}^2\tau_x')} \right]^{1/2} \right|^2 = \frac{1}{(1 + \tau_x'^2)^{1/2}(1 + \mathbf{e}^4\tau_x'^2)^{1/2}}, \quad (18)$$

and thus

$$h_m = \frac{\pi^{1/2}\mathbf{e}}{4\beta\xi_x} \int_{-\xi_x}^{\xi_x} [(1 + \tau_x'^2)(1 + \mathbf{e}^4\tau_x'^2)]^{-1/2} d\tau_x' = \frac{\pi^{1/2}}{2\beta\xi_x\mathbf{e}} \int_0^{\xi_x} [(1 + \tau_x'^2)(\mathbf{e}^{-4} + \tau_x'^2)]^{-1/2} d\tau_x'. \quad (19)$$

Equation (19) is now an incomplete elliptic integral of the first kind³¹:

$$F(\phi \setminus \alpha) = a \int_0^x [(t^2 + a^2)(t^2 + b^2)]^{-1/2} dt, \quad (20)$$

where $\tan \phi = x/b$, $\cos \alpha = b/a$, and $a > b$. In this case $a = 1$ and $b = 1/\mathbf{e}^2$ ($\mathbf{e} > 1$). The integral does not depend on B and increases monotonically with ellipticity toward the state in which h_m is optimized with respect to ξ_y and independent of ξ_x . We therefore write h_m in terms of ξ_y :

$$h_m = \frac{\pi^{1/2}}{2B\xi_y^{1/2}} F\left(\tan^{-1} \xi_y \setminus \cos^{-1} \frac{1}{\mathbf{e}^2}\right). \quad (21)$$

The optimal value of ξ_y in the limit of high B and \mathbf{e} is thus found to be 3.3189, to be compared with the limiting value of 1.3919 for $\mathbf{e} = 1$ and with the optimal value of 2.84 for the complete version of h_m when $B = 0$. We note that Eq. (21) is inconsistent with the expression quoted by Adams and Ferguson,²² which depends equally on w_x and w_y .

From our analytic function, with $B = 16$, $\mathbf{e} = 1$, and $\xi_x = \xi_y = 1$, we obtain $h_m = \pi^{3/2}/128 = 0.0435$. This result is in good agreement with the computed result of 0.0430, which is lower by 1%. In general, the approximation $\exp(-4s^2) = 1$ will make the analytic value an overestimate.

6. APPLICATION

To determine the best focusing parameters for a specific crystal and wavelength, it is first necessary to calculate the phase-matching and walk-off angles from refractive-index data such as Sellmeier coefficients. From the phase-matching angle and the refractive index can be calculated the crystal cut, for either normal or Brewster-angle incidence, and from the walk-off angle it is possible to work out the B coefficient. The conversion efficiency and focusing parameters are then given by reference to Figs. 1 and 2.

For harmonic generation from beams of circular cross section, elliptical focusing will rarely be worth the losses and alignment difficulties introduced by the additional lenses. If a circular Gaussian beam is focused into the crystal at Brewster's angle, however, the beam within the crystal will be elliptical, with $\mathbf{e} = \eta$. For negative uniaxial crystals the ellipticity will unfortunately lie in the wrong direction if the crystal is cut to minimize reflection of the fundamental, as when it is used within an enhancement cavity. With positive uniaxial crystals, however, the ellipticity introduced by the Brewster-angled face will be in the correct sense.

Ring enhancement cavities usually include off-axis spherical mirrors, and the modes that they support thus have elliptical foci. The appropriate orientation of the doubling crystal will in such cases often not only tolerate this ellipticity but allow increased efficiency to follow from it; the same is true for the frequency doubling of diode lasers. For frequency quadrupling,^{6,7} the efficiency depends quadratically on that of the first doubling stage, and elliptical focusing could thus improve the overall conversion efficiency by a factor of 2. In all cases care must be taken to ensure that the focal ellipticity and polarization are appropriately oriented.

The analysis presented in this paper is valid for all but the broadest-bandwidth lasers. Longitudinal coherence is required explicitly over a length $l_1 = (\Delta k/k)l$, which corresponds to the longitudinal range that contributes to the instantaneous output intensity; this length is the result of the Guoy shift and will thus be of the order of a wavelength. It is also necessary, however, that the phase-matching condition be satisfied for the instantaneous wavelength, and thus dispersion of the birefringence should introduce a phase mismatch

$$l\Delta k \sim l \frac{d(\Delta k)}{d\phi} \frac{d\phi}{d\lambda} \Delta\lambda \ll 1. \quad (22)$$

Hence [refer to Eq. (4)], the laser bandwidth is limited to $\Delta\lambda \sim 1/(lk_2\rho d\phi/d\lambda)$, where $d\phi/d\lambda$ is the sensitivity of tuning angle to wavelength. For a typical example of doubling green light in a 7-mm crystal of β -barium borate, this sensitivity is ~ 0.1 nm, making our results valid for most lasers with pulses longer than a few picoseconds.

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