SEMESTER 2 EXAMINATION 2017-2018

CLASSICAL MECHANICS

Duration: 120 MINS (2 hours)

This paper contains 9 questions.

Answers to Section A and Section B must be in separate answer books

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language dictionary is permitted ONLY IF it is a paper version of a direct ‘Word to Word’ translation dictionary AND it contains no notes, additions or annotations.

Section A

A1. Define the centre of mass coordinate $\mathbf{R}$ for a collection of particles with masses m_i, and show that the total momentum $\mathbf{P} = M\dot{\mathbf{R}}$, where M is the total mass. [4]

A2. State the *centre of mass* condition for an isolated system of particles with masses m_i and positions $\mathbf{r}_i = \boldsymbol{\rho}_i + \mathbf{R}$, where $\boldsymbol{\rho}_i$ are the positions relative to the centre of mass coordinate $\mathbf{R}$. [1]

Hence show that the total kinetic energy T of the particles may be written as

$$T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\sum_i m_i \dot{\boldsymbol{\rho}}_i^2,$$

where $M = \sum_i m_i$ is the combined mass of the system. [3]

A3. Consider a lone planet in an orbit, with non-zero eccentricity e, about a remote star. Sketch the orbit, labelling

(a) the semimajor and semiminor axes, [1]

(b) the position of the star, in terms of e, along the semimajor axis, [1]

(c) the points where the planet's radial velocity momentarily vanishes, and [1]

(d) the point where the planet's angular velocity is a maximum. [1]

A4. An evil biologist surmises that if a colony of ants is kept on a rotating turntable, the insects will develop and evolve so that their left legs are stronger than their right legs. Explain the physical rationale that the mad scientist might have for this phenomenon, and the direction in which her turntable rotates. [4]

A5. Show that the gravitational field due to a horizontal uniform thin disc of thickness d, radius R and density ρ, at a distance h vertically above the disc's centre, has a magnitude

$$2\pi G \rho d \left(1 - \frac{h}{\sqrt{R^2 + h^2}}\right),$$

where G is the gravitational constant. [4]

Section B

B1. (a) State the vector relationship between torque and angular momentum, and explain what happens when

(i) the applied torque vector is parallel to the angular momentum, and

(ii) the applied torque is at an angle to the angular momentum. [4]

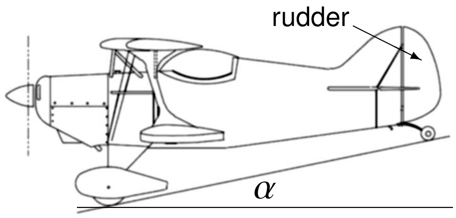
(b) Explain what is meant by an object's *moment of inertia* about an axis, and define it mathematically in terms of the distribution of the object's mass. [2]

(c) An aeroplane's propeller, shown above, has a diameter of 1.88 m and mass of 14.7 kg. Making clear any assumptions, estimate

(i) its moment of inertia about the axis of the central hole; and [4]

(ii) the torque required to increase its rate of rotation from 1200 to 2400 rpm (revolutions per minute) in 2 seconds. [2]

(d) During the take-off run, the tail of the aeroplane rises, so that the aeroplane rotates through an angle $\alpha = 12°$ to the attitude shown below.



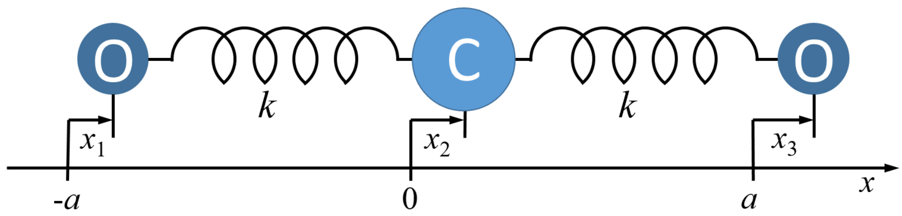
Explain and calculate the effect of changing the propeller's vector angular momentum. Assume that, as viewed by the pilot, the propeller rotates anticlockwise at 3000 rpm, and that the change of attitude takes 3 s. [4]

(e) To counteract this effect, the rudder deflects the airflow sideways. By considering momentum conservation, and taking the density of air to be 1.2 kg m^{-3}, estimate the cross-sectional area that the rudder must present to the airflow if it is 4.2 m from the propeller and the airspeed is 25 m s^{-1}. [4]

TURN OVER

B2. (a) Explain what is meant by (i) *simple harmonic motion* and (ii) the *normal mode* of an oscillating system. [4]

The stretch modes of the CO_2 molecule may be modelled by the classical system depicted below, in which the C and O atoms are rigid bobs with masses M_C and M_O, connected by two springs with spring constants k and constrained to move in only the x-direction. At rest, the O atoms are each separated from the C atom by a distance a, and x_1, x_2 and x_3 represent the displacements of the atoms from these rest positions.



(b) Setting out your working formally, derive the three equations of motion

$$\begin{aligned} M_O\ddot{x}_1 &= k(x_2 - x_1) \\ M_C\ddot{x}_2 &= k(x_3 - 2x_2 + x_1) \\ M_O\ddot{x}_3 &= k(x_2 - x_3) \end{aligned}$$

where $\ddot{x}_1 \equiv \mathrm{d}^2x_1/\mathrm{d}t^2$, etc. [4]

(c) By substituting the normal mode solutions $x_j = a_j\exp(\mathrm{i}\omega t)$, where $j = 1\ldots3$, show that the common frequency of motion ω must satisfy

$$\omega^2\,(M_O\omega^2 - k)\,\left[M_CM_O\omega^2 - (M_C{+}2M_O)k\right] = 0.$$ [8]

(d) Hence find expressions for the frequencies ω_{asym} and ω_{sym} of the *asymmetric stretch* and *symmetric stretch* modes, and comment upon how their ratio compares with the measured value of $\omega_{\mathrm{asym}}/\omega_{\mathrm{sym}} = 1.69$. [4]

The ratio of the atomic masses, M_O/M_C, is approximately $4/3$.

B3. (a) Explain what are meant by *centrifugal* and *Coriolis* forces. Outline their origins, the properties upon which they depend, and the directions in which they act. [4]

The crew cabin aboard the spaceship *Discovery One* occupies the inner rim of a cylinder 16 m in diameter, which rotates steadily to provide an apparent gravity close to that on Earth. A running track between the workstations and rest areas allows astronauts to exercise.

(b) Calculate the angular velocity with which the cylinder should rotate. [2]

(c) The crew members quickly realize that it is easier to run in one direction than the other. Explain this observation, and estimate the difference quantitatively for a running speed of 2.5 m s^{-1}. [3]

(d) The air within the rim of the cylinder is maintained at approximately the Earth's atmospheric density of $1.2\,\mathrm{kg\,m^{-3}}$. Assuming that the air is uniform in temperature and density, and rotates with the cylinder, find the pressure difference between the edge of the cylinder and its axis. [5]

(e) The cylinder and associated equipment have a total mass of 50 000 kg. Find the magnitude and direction of the gravitational acceleration that would be experienced at the rim if this mass were concentrated at the cylinder's axle. [3]

(f) Explain how your answer to (e) would differ qualitatively if the mass were instead uniformly distributed around the cylindrical rim? [3]

The gravitational constant $G = 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$.

TURN OVER

B4. A spacecraft follows an elliptical orbit about the Earth and, while its rocket engines are inactive, is subject only to the Earth's gravitational attraction.

(a) Explain why the vector angular momentum is conserved and why this means that the orbit lies in a plane. [4]

(b) What other quantity or quantities is/are conserved, and why? [2]

The spacecraft's orbit is initially in the equatorial plane of the Earth, but must be changed to an elliptical orbit of the same eccentricity in a polar plane (i.e. from a plane normal to the Earth's rotation axis to one containing it).

(c) At which point in the elliptical orbit can the change of orbit be accomplished most efficiently? Explain your answer. [3]

(d) The change of orbit is accomplished by firing the spacecraft's rocket motor for a time that is much less than the orbital period. How will the spacecraft's velocities $\mathbf{v}_i$ and $\mathbf{v}_f$ immediately before and after the impulse be related? [2]

(e) Show that, if at time t the spacecraft has a total mass $m(t)$ and ejects exhaust gas from its rocket motor with a relative velocity $\mathbf{u}$, then its velocity $\mathbf{v}(t)$ satisfies

$$m\,\mathrm{d}\mathbf{v} = -\mathbf{u}\,\mathrm{d}m.$$ [2]

(f) Hence, noting any assumptions, show that the initial and final velocities $\mathbf{v}_i$ and $\mathbf{v}_f$ are related to the initial and final masses m_i and m_f by

$$\mathbf{v}_f = \mathbf{v}_i + \mathbf{u}\ln\frac{m_i}{m_f}.$$ [2]

(g) The spacecraft has an empty mass of 950 kg and is initially in a nearly circular orbit of radius 7230 km. Find the minimum mass of fuel that must be burned if the exhaust gas leaves with a relative speed of 3120 m s^{-1}. [5]

The mass of the Earth may be taken to be 5.97×10^{24} kg, and the gravitational constant to be $G = 6.67 \times 10^{-11}\ \mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$.

END OF PAPER