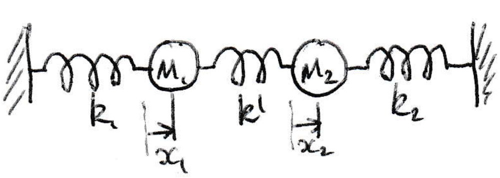
Avoided Crossings.

Consider two oscillators (m_1, k_1) and (m_2, k_2), coupled by a third spring k'



If m_2 were fixed, m_1 would oscillate with natural frequency $\omega_1 = \sqrt{\dfrac{k_1+k'}{m_1}}$

If m_1 were fixed, m_2 would oscillate with natural frequency $\omega_2 = \sqrt{\dfrac{k_2+k'}{m_2}}$

The general case is solved as before:

$$m_1\ddot{x}_1 = -k_1x_1 + k'(x_2-x_1)$$
$$m_2\ddot{x}_2 = -k'(x_2-x_1) - k_2x_2$$

$$\Rightarrow \begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix} = \begin{pmatrix}-(k_1+k')/m_1 & k'/m_1\\ k'/m_2 & -(k_2+k')/m_2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}-\omega_1^2 & k'/m_1\\ k'/m_2 & -\omega_2^2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

Look for solutions of the form $\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}A_1\\ A_2\end{pmatrix}\exp i\omega t$: substitute into equation above

$$\Rightarrow -\omega^2\begin{pmatrix}A_1\\ A_2\end{pmatrix}\exp i\omega t = \begin{pmatrix}-\omega_1^2 & k'/m_1\\ k'/m_2 & -\omega_2^2\end{pmatrix}\begin{pmatrix}A_1\\ A_2\end{pmatrix}\exp i\omega t$$

$$\Rightarrow \begin{pmatrix}\omega^2-\omega_1^2 & k'/m_1\\ k'/m_2 & \omega^2-\omega_2^2\end{pmatrix}\begin{pmatrix}A_1\\ A_2\end{pmatrix} = 0 \qquad (1)$$

$$\Rightarrow \begin{vmatrix}\omega^2-\omega_1^2 & k'/m_1\\ k'/m_2 & \omega^2-\omega_2^2\end{vmatrix} = 0$$

$$\Rightarrow (\omega^2-\omega_1^2)(\omega^2-\omega_2^2) - k'^2/m_1m_2 = 0$$

$$\Rightarrow (\omega^2)^2 - (\omega_1^2+\omega_2^2)(\omega^2) + (\omega_1^2\omega_2^2 - k'^2/m_1m_2) = 0$$

$$\Rightarrow \omega^2 = \frac{\omega_1^2+\omega_2^2 \pm \sqrt{(\omega_1^2+\omega_2^2)^2 - 4\{\omega_1^2\omega_2^2 - k'^2/m_1m_2\}}}{2} = \frac{\omega_1^2+\omega_2^2}{2} \pm \sqrt{\left(\frac{\omega_1^2-\omega_2^2}{2}\right)^2 + \frac{k'^2}{m_1m_2}}$$

Note that (a) if coupling is weak enough to neglect k', $\omega^2 = \omega_1^2$ or ω_2^2

(b) if $\omega_1 = \omega_2$, $\omega^2 = \omega_1^2 \pm k'/\sqrt{m_1 m_2} = \frac{k_1+k'}{m_1} \pm \frac{k'}{\sqrt{m_1 m_2}}$

$\Rightarrow$ if $m_1 = m_2$, $\omega^2 = \frac{k_1}{m_1}$ or $\frac{k_1}{m_1} + 2\frac{k'}{m_1}$

(a) corresponds to independent oscillators with the natural frequencies determined initially;
(b) corresponds to the case previously considered, where $\omega^2 = \frac{k_1}{m_1}$ for the symmetric mode,
$\omega^2 = \frac{k_1+2k'}{m_1}$ for the antisymmetric mode

Substituting ω^2 into (1) to determine the eigenvectors,

$$\begin{pmatrix} \frac{\omega_2^2-\omega_1^2}{2} \pm \sqrt{\left(\frac{\omega_2^2-\omega_1^2}{2}\right)^2 + \frac{k'^2}{m_1 m_2}} & \frac{k'}{m_1} \\ \frac{k'}{m_2} & \frac{\omega_1^2-\omega_2^2}{2} \pm \sqrt{\left(\frac{\omega_1^2-\omega_2^2}{2}\right)^2 + \frac{k'^2}{m_1 m_2}} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

Since the determinant is zero, both equations from this give the same solutions. We therefore consider just one, eg the top,

$$\left[\frac{\omega_2^2-\omega_1^2}{2} \pm \sqrt{\left(\frac{\omega_2^2-\omega_1^2}{2}\right)^2 + \frac{k'^2}{m_1 m_2}}\right] A_1 + \frac{k'}{m_1} A_2 = 0$$

$$\Rightarrow \frac{A_2}{A_1} = -\frac{\frac{\omega_2^2-\omega_1^2}{2} \pm \sqrt{\left(\frac{\omega_2^2-\omega_1^2}{2}\right)^2 + \frac{k'^2}{m_1 m_2}}}{k'/m_1} = -\frac{\omega^2-\omega_1^2}{k'/m_1}$$

The energy in each oscillator will be $\frac{1}{2} m_1 A_1^2 \omega^2$, $\frac{1}{2} m_2 A_2^2 \omega^2$

$\Rightarrow$ fraction of energy in oscillator 1 will be $\frac{m_1 A_1^2}{m_1 A_1^2 + m_2 A_2^2} = \frac{1}{1 + \frac{m_2}{m_1}\left(\frac{A_2}{A_1}\right)^2} = \mathcal{E}_1$

$$\Rightarrow \mathcal{E}_1 = \left\{1 + \frac{m_2}{m_1}\left(\frac{m_1}{k'}\right)^2 \left[\frac{\omega_2^2-\omega_1^2}{2} \pm \sqrt{\left(\frac{\omega_2^2-\omega_1^2}{2}\right)^2 + \frac{k'^2}{m_1 m_2}}\right]^2\right\}^{-1}$$

$$= \left\{1 + \frac{m_1 m_2}{k'^2}\left[\left(\frac{\omega_2^2-\omega_1^2}{2}\right)^2 + \left(\frac{\omega_2^2-\omega_1^2}{2}\right)^2 + \frac{k'^2}{m_1 m_2} \pm (\omega_2^2-\omega_1^2)\sqrt{\left(\frac{\omega_2^2-\omega_1^2}{2}\right)^2 + \frac{k'^2}{m_1 m_2}}\right]\right\}^{-1}$$

$$= \frac{1}{2}\left\{1 + \frac{m_1 m_2}{k'^2}\left[\left(\frac{\omega_2^2-\omega_1^2}{2}\right)^2 \pm \left(\frac{\omega_2^2-\omega_1^2}{2}\right)\sqrt{\left(\frac{\omega_2^2-\omega_1^2}{2}\right)^2 + \frac{k'^2}{m_1 m_2}}\right]\right\}^{-1}$$

$$\Rightarrow \mathcal{E}_1 = \left\{ 1 + \frac{m_2}{m_1}\left(\frac{\omega^2-\omega_1^2}{k'/m_1}\right)^2 \right\}^{-1}$$

$$= \left\{ 1 + \frac{\left(\omega^2-\omega_1^2\right)^2}{k'^2/m_1m_2} \right\}^{-1}$$

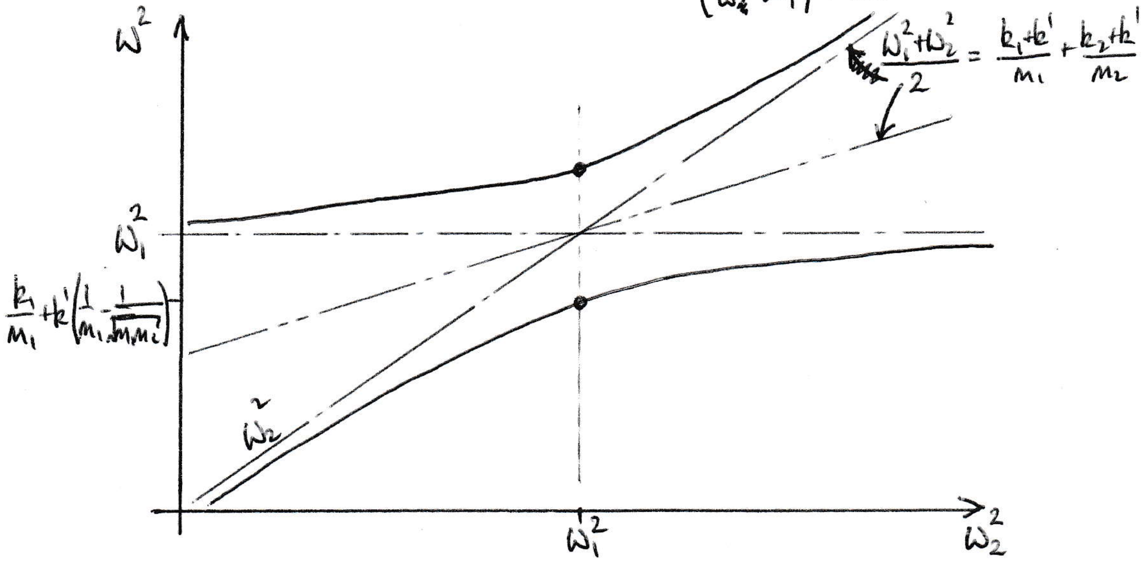
$$\Rightarrow \mathcal{E}_1 = \frac{k'^2/m_1m_2}{\left(\omega^2-\omega_1^2\right)^2 + k'^2/m_1m_2}.$$

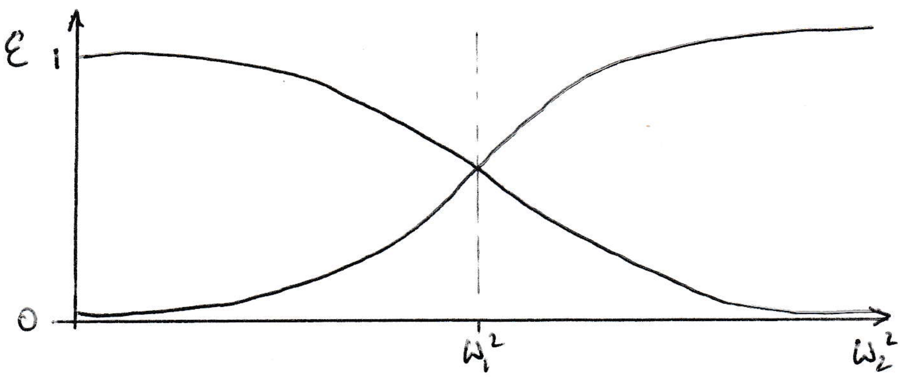
$\mathcal{E}_2$ is then given since $\mathcal{E}_1+\mathcal{E}_2 = 1$.

If we write $\Omega^2 = \dfrac{k'}{\sqrt{m_1m_2}}$,

$$\omega^2 = \frac{\omega_1^2+\omega_2^2}{2} \pm \sqrt{\left(\frac{\omega_1^2-\omega_2^2}{2}\right)^2 + \Omega^4}$$

$$\mathcal{E}_1 = \frac{\Omega^4}{\left(\omega_2^2-\omega_1^2\right)^2 + \Omega^4}.$$





- for $\omega_1^2-\omega_2^2 \gg \Omega^2$, $\mathcal{E}_{1,2} = 0,1$: dephasing too rapid for significant population/energy transfer
  $\Rightarrow$ oscillators effectively decoupled.
- adiabatic passage: sweep ω_2 from one side of resonance to other: population remains in eigenstate
  $\Rightarrow$ 100% transfer from one oscillator to the other.

See Steven G. Johnson, "When stationary modes aren't stationary..." math.mit.edu/~stevenj/coupling.pdf