Rotating Coordinate Systems.

- obvious local coordinates — <u>not</u> an inertial frame
- "fictitious" forces: centrifugal
  Coriolis
- projectile trajectories; Foucault's pendulum; weather systems; ocean currents

Recall result that for a vector A of fixed length, rotating about origin with angular velocity $\underline{\omega}$,

$$\frac{d\underline{A}}{dt} = \underline{\omega} \times A. \qquad (2.1)\ \text{p10.}$$

Apply this to the unit vectors $\hat{\underline{i}}', \hat{\underline{j}}', \hat{\underline{k}}'$ of a rotating frame:

$$\frac{d\hat{\underline{i}}'}{dt} = \underline{\omega} \times \hat{\underline{i}}' \qquad \text{etc.}$$

Consider an <u>arbitrary</u> vector $\underline{a}$ and express it in terms of unit vectors of an inertial frame $\{\hat{\underline{i}}\hat{\underline{j}}\hat{\underline{k}}\}$ and the rotating frame $\{\hat{\underline{i}}'\hat{\underline{j}}'\hat{\underline{k}}'\}$:

$$\underline{a} = \underbrace{a_i\hat{\underline{i}} + a_j\hat{\underline{j}} + a_k\hat{\underline{k}}}_{\text{inertial}} = \underbrace{a_i'\hat{\underline{i}}' + a_j'\hat{\underline{j}}' + a_k'\hat{\underline{k}}'}_{\text{eg. northings, eastings, altitude}}$$

Now find equations of motion for 'vector' $\begin{pmatrix} a_i' \\ a_j' \\ a_k' \end{pmatrix}$, distinct from $a_i'\hat{\underline{i}}' + a_j'\hat{\underline{j}}' + a_k'\hat{\underline{k}}'$:

Differentiate $\underline{a}$ wrt t, noting that $\hat{\underline{i}}'\hat{\underline{j}}'\hat{\underline{k}}'$ vary:

$$
\begin{aligned}
\frac{d\underline{a}}{dt} &= \frac{da_i}{dt}\hat{\underline{i}} + \frac{da_j}{dt}\hat{\underline{j}} + \frac{da_k}{dt}\hat{\underline{k}} \\
&= \frac{da_i'}{dt}\hat{\underline{i}}' + \frac{da_j'}{dt}\hat{\underline{j}}' + \frac{da_k'}{dt}\hat{\underline{k}}' + a_i'\frac{d\hat{\underline{i}}'}{dt} + a_j'\frac{d\hat{\underline{j}}'}{dt} + a_k'\frac{d\hat{\underline{k}}'}{dt} \\
&= \frac{da_i'}{dt}\hat{\underline{i}}' + \frac{da_j'}{dt}\hat{\underline{j}}' + \frac{da_k'}{dt}\hat{\underline{k}}' + a_i'(\underline{\omega}\times\hat{\underline{i}}') + a_j'(\underline{\omega}\times\hat{\underline{j}}') + a_k'(\underline{\omega}\times\hat{\underline{k}}') \\
&= \frac{da_i'}{dt}\hat{\underline{i}}' + \frac{da_j'}{dt}\hat{\underline{j}}' + \frac{da_k'}{dt}\hat{\underline{k}}' + \underline{\omega}\times\left(a_i'\hat{\underline{i}}' + a_j'\hat{\underline{j}}' + a_k'\hat{\underline{k}}'\right) \\
&= \frac{da_i'}{dt}\hat{\underline{i}}' + \frac{da_j'}{dt}\hat{\underline{j}}' + \frac{da_k'}{dt}\hat{\underline{k}}' + \underline{\omega}\times\underline{a}.
\end{aligned}
$$

Differentiate a second time:

$$
\begin{aligned}
\frac{d^2\underline{a}}{dt^2} &= \frac{d^2a_i'}{dt^2}\hat{\underline{i}}' + \frac{d^2a_j'}{dt^2}\hat{\underline{j}}' + \frac{d^2a_k'}{dt^2}\hat{\underline{k}}' + \frac{da_i'}{dt}\frac{d\hat{\underline{i}}'}{dt} + \frac{da_j'}{dt}\frac{d\hat{\underline{j}}'}{dt} + \frac{da_k'}{dt}\frac{d\hat{\underline{k}}'}{dt} + \underline{\omega}\times\frac{d\underline{a}}{dt} \\
&= \frac{d^2a_i'}{dt^2}\hat{\underline{i}}' + \frac{d^2a_j'}{dt^2}\hat{\underline{j}}' + \frac{d^2a_k'}{dt^2}\hat{\underline{k}}' + \underline{\omega}\times\left(\frac{da_i'}{dt}\hat{\underline{i}}' + \frac{da_j'}{dt}\hat{\underline{j}}' + \frac{da_k'}{dt}\hat{\underline{k}}'\right) \\
&\quad + \underline{\omega}\times\left(\frac{da_i'}{dt}\hat{\underline{i}}' + \frac{da_j'}{dt}\hat{\underline{j}}' + \frac{da_k'}{dt}\hat{\underline{k}}' + \underline{\omega}\times\underline{a}\right)
\end{aligned}
$$

$$
\Rightarrow \underline{F} = m\frac{d^2\underline{a}}{dt^2} = m\left\{\underbrace{\left(\frac{d^2a_i'}{dt^2}\hat{\underline{i}}' + \frac{d^2a_j'}{dt^2}\hat{\underline{j}}' + \frac{d^2a_k'}{dt^2}\hat{\underline{k}}'\right)}_{\text{apparent acceleration}} + 2\underline{\omega}\times\left[\underbrace{\left(\frac{da_i'}{dt}\hat{\underline{i}}' + \frac{da_j'}{dt}\hat{\underline{j}}' + \frac{da_k'}{dt}\hat{\underline{k}}'\right)}_{\text{apparent velocity}} + \underline{\omega}\times\underbrace{\underline{a}}_{\text{apparent = real position}}\right]\right\}
$$

$\Rightarrow$ if $\underline{r}$ is apparent position in rotating frame, defined by $\begin{pmatrix} a_i' \\ a_j' \\ a_k' \end{pmatrix}$,

$$
\underline{F} = m\left\{\ddot{\underline{r}} + 2\underline{\omega}\times\dot{\underline{r}} + \underline{\omega}\times(\underline{\omega}\times\underline{r})\right\}
$$

i.e.

$$
m\ddot{\underline{r}} = \underline{F} \underbrace{- 2m\underline{\omega}\times\dot{\underline{r}}}_{\text{Coriolis}} \underbrace{- m\underline{\omega}\times(\underline{\omega}\times\underline{r})}_{\text{centrifugal}}.
$$