1

Motion of Systems of Particles

This chapter contains formal arguments showing (i) that the total external force acting on a system of particles is equal to the rate of change of its total linear momentum and (ii) that the total external torque acting is equal to the rate of change of the total angular momentum. Although you should ensure you understand the arguments, the important point is the simple and useful general results which emerge.

1.1 Linear Motion

Consider a system of N particles labelled $1, 2, \ldots, N$ with masses m_i at positions $\mathbf{r}_i$. Let the momentum of the ith particle be $\mathbf{p}_i$. The total force acting and the total linear momentum are

$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i \quad \text{and} \quad \mathbf{P} = \sum_{i=1}^{N} \mathbf{p}_i,$$

respectively. Summing the equations of motion, $\mathbf{F}_i = \dot{\mathbf{p}}_i$ (Newton's second law), for all the particles immediately leads to

$$\mathbf{F} = \dot{\mathbf{P}}.$$

To make this more useful, we divide up the force $\mathbf{F}_i$ on the ith particle into the external force plus the sum of all the internal forces due to the other particles:

$$\mathbf{F}_i = \mathbf{F}_i^{\mathrm{ext}} + \sum_{j \neq i} \mathbf{F}_{ij}.$$

Here, $\mathbf{F}_{ij}$ is the force on the ith particle due to the jth. The payoff for using this decomposition is that the internal forces are related in pairs by Newton's third law,

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji},$$

and therefore,

$$\mathbf{F} = \sum_{i=1}^{N} \left(\mathbf{F}_i^{\mathrm{ext}} + \sum_{j \neq i} \mathbf{F}_{ij} \right) = \sum_{i=1}^{N} \mathbf{F}_i^{\mathrm{ext}} + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \mathbf{F}_{ij}.$$

The first term on the RHS is simply the total external force, $\mathbf{F}^{\mathrm{ext}}$, and the second term vanishes because the internal forces cancel in pairs. Thus we end up with the result:

$$\boxed{\mathbf{F}^{\mathrm{ext}} = \dot{\mathbf{P}}}. \tag{1.1}$$

- The total external force is equal to the rate of change of the total linear momentum of the system.
- We used Newton's third law to cancel the internal forces in pairs.
- If the external force vanishes, $F^{\text{ext}} = 0$, then $\dot{\mathbf{P}} = 0$, so $\mathbf{P}$ is constant and we can state:

The linear momentum of a system subject to no net external force is conserved.

1.1.1 Centre of Mass

Define the centre of mass, $\mathbf{R}$, by,

$$\mathbf{R} = \frac{\sum_{i=1}^{N} m_i \mathbf{r}_i}{\sum_{i=1}^{N} m_i} = \frac{1}{M} \sum_{i=1}^{N} m_i \mathbf{r}_i,$$

where $M = \sum m_i$ is the total mass.

If the individual masses are constant, then the velocity of the centre of mass is found from,

$$M\dot{\mathbf{R}} = \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i = \mathbf{P}.$$

Furthermore, we just saw above that $\mathbf{F}^{\text{ext}} = d\mathbf{P}/dt$. So we have the following results:

$$\boxed{\mathbf{P} = M\dot{\mathbf{R}}} \quad \text{and} \quad \boxed{\mathbf{F}^{\text{ext}} = M\ddot{\mathbf{R}}}. \tag{1.2}$$

- In the *absence* of a net external force, the centre of mass moves with constant velocity. This says (once again) that:

The linear momentum of a system subject to no net external force is conserved.

- If the net external force is non-zero, the centre of mass moves as if the total mass of the system were there, acted on by the total external force.

It is often useful to look at the system of particles with positions measured relative to the centre of mass. If $\boldsymbol{\rho}_i$ is the location of the ith particle with respect to the centre of mass then (see figure 1.1),

$$\boxed{\mathbf{r}_i = \mathbf{R} + \boldsymbol{\rho}_i}. \tag{1.3}$$

1.1.2 Kinetic Energy of a System of Particles

Let's look at the total kinetic energy T of the system using the decomposition in equation (1.3).

$$\begin{aligned} T &= \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 = \sum_{i=1}^{N} \frac{1}{2} m_i (\dot{\mathbf{R}} + \dot{\boldsymbol{\rho}}_i)^2 \\ &= \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\mathbf{R}}^2 + \sum_{i=1}^{N} m_i \dot{\boldsymbol{\rho}}_i \cdot \dot{\mathbf{R}} + \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\boldsymbol{\rho}}_i^2. \end{aligned}$$

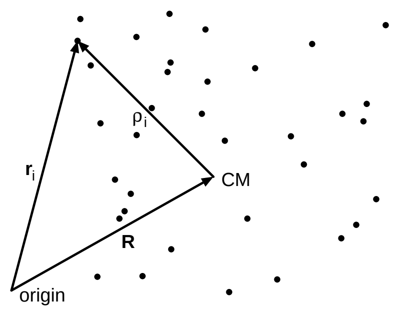

Figure 1.1 Particle positions measured with respect to the Centre of Mass

The second term on the RHS vanishes since $\sum m_i \boldsymbol{\rho}_i = 0$ and $\sum m_i \dot{\boldsymbol{\rho}}_i = 0$ by the definition of the centre of mass. This leaves,

$$T = \frac{1}{2} M \dot{\mathbf{R}}^2 + \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\boldsymbol{\rho}}_i^2,$$

which we write as,

$$\boxed{T = \frac{1}{2} M \dot{\mathbf{R}}^2 + T_{\text{CM}}}. \tag{1.4}$$

The total kinetic energy has one term from the motion of the centre of mass and a second term from the kinetic energy of motion with respect to the centre of mass. Since particle velocities are different when measured in different inertial reference frames, the kinetic energy will in general be different in different frames. However, T_{CM}, the kinetic energy with respect to the center of mass is the *same* in all inertial frames and is an "internal" kinetic energy of the system (the sum of T_{CM} and the potential energy due to the internal interactions is the total internal energy, U, as used in thermodynamics). To prove this, note that a Galilean transformation from a frame S to a frame S' moving at velocity $\mathbf{v}$ with respect to S changes particle positions by:

$$\mathbf{r}_i \to \mathbf{r}'_i = \mathbf{r}_i - \mathbf{v}t.$$

The centre of mass transforms similarly,

$$\mathbf{R} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i} \to \mathbf{R}' = \frac{\sum m_i \mathbf{r}'_i}{\sum m_i} = \mathbf{R} - \mathbf{v}t,$$

so that positions and velocities with respect to the centre of mass are *unchanged*:

$$\begin{array}{lllllllll} \boldsymbol{\rho}'_i & = & \mathbf{r}'_i - \mathbf{R}' & = & (\mathbf{r}_i - \mathbf{v}t) - (\mathbf{R} - \mathbf{v}t) & = & \mathbf{r}_i - \mathbf{R} & = & \boldsymbol{\rho}_i \\ \dot{\boldsymbol{\rho}}'_i & = & \dot{\mathbf{r}}'_i - \dot{\mathbf{R}}' & = & (\dot{\mathbf{r}}_i - \mathbf{v}) - (\dot{\mathbf{R}} - \mathbf{v}) & = & \dot{\mathbf{r}}_i - \dot{\mathbf{R}} & = & \dot{\boldsymbol{\rho}}_i \end{array}$$

The decomposition of the kinetic energy in equation (1.4) can be useful in problem solving. For example, if a ball rolls down a ramp, you can express the kinetic energy as a sum of one term coming from the linear motion of the centre of mass plus another term for the rotational motion about the centre of mass (the kinetic energy of rotational motion is discussed further later in the notes).

System of Two Particles Now apply the kinetic energy expression in equation (1.4) to a system of two particles. Write the particle velocities as $\mathbf{u}_1 = \dot{\mathbf{r}}_1$ and $\mathbf{u}_2 = \dot{\mathbf{r}}_2$, so that:

$$\mathbf{u}_1 = \dot{\mathbf{R}} + \dot{\boldsymbol{\rho}}_1 \qquad \text{and} \qquad \mathbf{u}_2 = \dot{\mathbf{R}} + \dot{\boldsymbol{\rho}}_2.$$

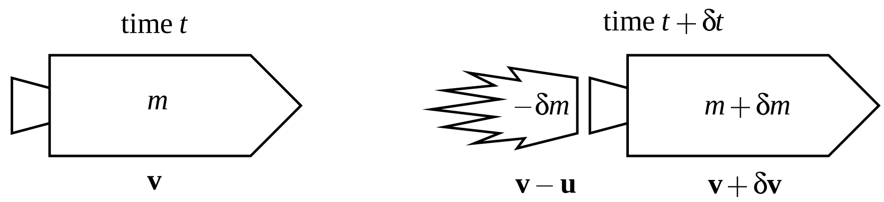

Figure 1.2 Motion of a rocket. We consider the rocket at two closely spaced instants of time, t and $t+\delta t$.

Subtracting these two equations gives $\mathbf{u}_1-\mathbf{u}_2=\dot{\boldsymbol{\rho}}_1-\dot{\boldsymbol{\rho}}_2$, while the centre of mass condition states that $m_1\dot{\boldsymbol{\rho}}_1+m_2\dot{\boldsymbol{\rho}}_2=0$. We can thus solve for $\dot{\boldsymbol{\rho}}_1$ and $\dot{\boldsymbol{\rho}}_2$:

$$\dot{\boldsymbol{\rho}}_1=\frac{m_2(\mathbf{u}_1-\mathbf{u}_2)}{m_1+m_2},\qquad \dot{\boldsymbol{\rho}}_2=\frac{-m_1(\mathbf{u}_1-\mathbf{u}_2)}{m_1+m_2}.$$

Substituting these in the kinetic energy expression gives,

$$T=\frac{1}{2}(m_1+m_2)\dot{\mathbf{R}}^2+\frac{1}{2}\frac{m_1m_2}{m_1+m_2}(\mathbf{u}_1-\mathbf{u}_2)^2.$$

The quantity $m_1m_2/(m_1+m_2)$ appearing here is called the *reduced mass*. We will meet it again (briefly) in chapter 3 on Kepler's laws.

1.1.3 Examples

Rocket Motion We can use our results for the motion of a system of particles to describe so-called "variable mass" problems, where the mass of the (part of) the system we are interested in changes with time. A prototypical example is the motion of a rocket in deep space. The rocket burns fuel and ejects the combustion products at high speed (relative to the rocket), thereby propelling itself forward. To describe this quantitatively, we refer to the diagram in figure 1.2 and proceed as follows.

We consider the rocket at two closely spaced instants of time. At time t the rocket and its remaining fuel have mass m and velocity $\mathbf{v}$. In a short additional interval δt the rocket's mass changes to $m+\delta m$ as it burns a mass $-\delta m$ of fuel (note that δm is *negative* since the rocket uses up fuel for propulsion) and the rocket's velocity changes to $\mathbf{v}+\delta\mathbf{v}$. The exhaust gases are ejected with velocity $-\mathbf{u}$ with respect to the rocket, which is velocity $\mathbf{v}-\mathbf{u}$ with respect to an external observer. Hence, at time $t+\delta t$ we have a rocket of mass $m+\delta m$ moving with velocity $\mathbf{v}+\delta\mathbf{v}$ together with a mass $-\delta m$ of gas with velocity $\mathbf{v}-\mathbf{u}$.

If the rocket is in deep space, far from any stars or planets, there is no gravitational force or other external force on the system, so its overall linear momentum is conserved. Therefore, we may equate the linear momentum of the system at times t and $t+\delta t$,

$$m\mathbf{v}=(m+\delta m)(\mathbf{v}+\delta\mathbf{v})-\delta m(\mathbf{v}-\mathbf{u}).$$

Cancelling terms we find,

$$\mathbf{u}\delta m+m\delta\mathbf{v}+\delta m\delta\mathbf{v}=0.$$

We take the limit $\delta t\to 0$, so that the $\delta m\delta\mathbf{v}$ term, which is second order in infinitesimal quantities, drops out, leaving:

$$\mathbf{u}\frac{dm}{m}=-d\mathbf{v}.$$

If the rocket initially has velocity $\mathbf{v}_i$ when its mass is m_i, and ends up with velocity $\mathbf{v}_f$ when its mass is m_f, we integrate this equation to find:

$$\boxed{\mathbf{v}_f = \mathbf{v}_i + \mathbf{u}\ln\left(\frac{m_i}{m_f}\right)}. \tag{1.5}$$

The fact that the increase in the rocket's speed depends logarithmically on the ratio of initial and final masses is the reason why rockets are almost entirely made up of fuel when they are launched (the function $\ln x$ grows *very* slowly with x). It also explains why multi-stage rockets are advantageous: once you have burnt up some fuel, you don't want to carry around the structure that contained it, since this will reduce the ratio m_i/m_f for the subsequent motion.

Rope Falling Onto a Table Here we'll consider a system where an external force acts. A flexible rope with mass per unit length ρ is suspended just above a table. The rope is released from rest. Find the force on the table when a length x of the rope has fallen to the table.

Our system here is the rope. The external forces in the vertical direction are the weight of the rope, ρag, acting downwards plus an upward normal force F exerted on the rope by the tabletop. We want to determine F.

The rope falls freely onto the table, so its downward acceleration is g. If we let $v = \dot{x}$, this means that $\dot{v} = g$ and $v^2 = 2gx$.

Suppose that a length x of the rope has reached the table top after time t, when the speed of the falling section is v. A short time δt later, the length of rope on the table is $x + \delta x$ and the speed of the falling section is $v + \delta v$. The downward components of the system's total momentum at times t and $t + \delta t$ are therefore:

$$\begin{aligned} p(t) &= \rho(a-x)v, \\ p(t+\delta t) &= \rho(a-x-\delta x)(v+\delta v). \end{aligned}$$

Working to first order in small quantities,

$$\delta p = p(t+\delta t) - p(t) = \rho(a-x)\delta v - \rho v \delta x.$$

Taking the limit $\delta t \to 0$, we find that the rate of change of momentum is,

$$\frac{dp}{dt} = \rho(a-x)\dot{v} - \rho v\dot{x} = \rho(a-x)g - 2\rho xg.$$

Therefore, equating the external force to the rate of change of momentum gives,

$$\rho ag - F = \rho(a-x)g - 2\rho xg,$$

or finally,

$$F = 3\rho xg.$$

1.2 Angular Motion

The angular equation of motion for each particle is

$$\mathbf{r}_i \times \mathbf{F}_i = \frac{d}{dt}(\mathbf{r}_i \times \mathbf{p}_i).$$

The total angular momentum of the system and the total torque acting are:

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i \quad \text{and} \quad \boldsymbol{\tau} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i$$

As before we split the total force on each particle into external and internal parts. We then make a corresponding split in the total torque:

$$\begin{aligned} \boldsymbol{\tau} &= \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} + \sum_{i=1}^{N} \mathbf{r}_i \times \sum_{j \neq i} \mathbf{F}_{ij} \\ &\equiv \boldsymbol{\tau}^{\text{ext}} + \boldsymbol{\tau}^{\text{int}}. \end{aligned}$$

Recall that in the linear case, we were able to cancel the internal forces in pairs, because they satisfied Newton's third law. What is the corresponding result here? In other words, when can we ignore $\boldsymbol{\tau}^{\text{int}}$? To answer this, decompose $\boldsymbol{\tau}^{\text{int}}$ as follows,

$$\begin{aligned} \boldsymbol{\tau}^{\text{int}} &= \mathbf{r}_1 \times (\mathbf{F}_{12} + \mathbf{F}_{13} + \cdots + \mathbf{F}_{1N}) \\ &\quad + \mathbf{r}_2 \times (\mathbf{F}_{21} + \mathbf{F}_{23} + \cdots + \mathbf{F}_{2N}) + \cdots \\ &= (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F}_{12} + (\text{other pairs}). \end{aligned}$$

We have used Newton's third law to obtain the last line.

Now, if the internal forces act along the lines joining the particle pairs, then all the terms $(\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij}$ vanish and $\boldsymbol{\tau}^{\text{int}} = 0$. Thus $\boldsymbol{\tau}^{\text{int}} = 0$ for *central* internal forces. Examples are gravity and the Coulomb force.

With this proviso we obtain the result,

$$\sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} = \frac{d}{dt} \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i,$$

which is rewritten as,

$$\boxed{\boldsymbol{\tau}^{\text{ext}} = \dot{\mathbf{L}}}.$$

- This result applies when we use coordinates in an inertial frame (one in which Newton's laws apply).
- Note that we used both Newton's third law and the condition that the forces between particles were central in order to reach our result.

1.2.1 Angular Motion About the Centre of Mass

We will now see that taking moments about the centre of mass also leads to a simple result. To do this, look at the total angular momentum using the centre of mass coordinates:

$$\begin{aligned} \mathbf{L} &= \sum_{i=1}^{N} \mathbf{r}_i \times m_i \dot{\mathbf{r}}_i = \sum_{i=1}^{N} (\mathbf{R} + \boldsymbol{\rho}_i) \times m_i (\dot{\mathbf{R}} + \dot{\boldsymbol{\rho}}_i) \\ &= \sum_{i=1}^{N} \mathbf{R} \times m_i \dot{\mathbf{R}} + \sum_{i=1}^{N} \mathbf{R} \times m_i \dot{\boldsymbol{\rho}}_i + \sum_{i=1}^{N} \boldsymbol{\rho}_i \times m_i \dot{\mathbf{R}} + \sum_{i=1}^{N} \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i. \end{aligned}$$

The second and third terms on the RHS vanish since $\sum m_i \boldsymbol{\rho}_i = 0$ and $\sum m_i \dot{\boldsymbol{\rho}}_i = 0$ by the definition of the centre of mass. This leaves,

$$\mathbf{L} = \mathbf{R} \times M\dot{\mathbf{R}} + \sum_{i=1}^{N} \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i,$$

which we write as,

$$\boxed{\mathbf{L} = \mathbf{R} \times M\dot{\mathbf{R}} + \mathbf{L}_{\mathrm{CM}}}. \tag{1.6}$$

The total angular momentum therefore has two terms, which can be interpreted as follows. The first arises from the motion of the centre of mass about the origin of coordinates: this is called the *orbital* angular momentum and takes different values in different inertial frames. The second term, $\mathbf{L}_{\mathrm{CM}}$, arises from the angular motion about (relative to) the centre of mass (think of the example of a spinning planet orbiting the Sun): this is the *same* in all inertial frames and is an *intrinsic* or *spin* angular momentum (the proof of this is like the one given for T_{CM}, the kinetic energy relative to the CM, below equation (1.4) on page 3).

Finally, we take the time derivative of the last equation to obtain,

$$\begin{aligned}
\frac{d\mathbf{L}_{\mathrm{CM}}}{dt} &= \frac{d\mathbf{L}}{dt} - \mathbf{R} \times M\ddot{\mathbf{R}} \quad = \quad \boldsymbol{\tau}^{\mathrm{ext}} - \mathbf{R} \times \mathbf{F}^{\mathrm{ext}} \\
&= \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i^{\mathrm{ext}} - \sum_{i=1}^{N} \mathbf{R} \times \mathbf{F}_i^{\mathrm{ext}} \\
&= \sum_{i=1}^{N} (\mathbf{r}_i - \mathbf{R}) \times \mathbf{F}_i^{\mathrm{ext}} \\
&= \sum_{i=1}^{N} \boldsymbol{\rho}_i \times \mathbf{F}_i^{\mathrm{ext}} \quad \equiv \quad \boldsymbol{\tau}_{\mathrm{CM}}^{\mathrm{ext}}.
\end{aligned}$$

So we've found two results we can use when considering torques applied to a system:

$$\boxed{\boldsymbol{\tau}^{\mathrm{ext}} = \dot{\mathbf{L}}} \qquad \text{and} \qquad \boxed{\boldsymbol{\tau}_{\mathrm{CM}}^{\mathrm{ext}} = \dot{\mathbf{L}}_{\mathrm{CM}}}. \tag{1.7}$$

- These two equations say you can take moments either about the origin of an inertial frame, or about the centre of mass (even if the centre of mass is itself accelerating).

- Furthermore, in either case:

The angular momentum of a system subject to no external torque is constant.

1.3 Commentary

In deriving the general results above we assumed the validity of Newton's third law, so that we could cancel internal forces in pairs. We also assumed that the forces were central so that we could cancel internal torques in pairs. The assumption of central internal forces is very strong and we know of examples, such as the electromagnetic forces between moving particles, which are *not* central.

All we actually require is the validity of the results in equations (1.1) and (1.7). It is perhaps better to regard them as basic assumptions whose justification is that their consequences agree with experiment.

For the puzzle associated with the electromagnetic forces mentioned above, the resolution is that you have to ascribe energy, momentum and angular momentum to the electromagnetic field itself.