# 2

# Rotational Motion of Rigid Bodies

## 2.1 Rotations and Angular Velocity

A rotation $R(\hat{\mathbf{n}}, \theta)$ is specified by an axis of rotation, defined by a unit vector $\hat{\mathbf{n}}$ (2 parameters) and an angle of rotation $\theta$ (one parameter). Since you have a direction and a magnitude, you might suspect that rotations could be represented in some way by vectors. However, rotations through finite angles are *not* vectors, because they do not commute when you "add" or combine them by performing different rotations in succession. This is illustrated in figure 2.1

*Infinitesimal* rotations *do* commute when you combine them, however. To see this, consider a vector $\mathbf{A}$ which is rotated through an infinitesimal angle $d\phi$ about an axis $\hat{\mathbf{n}}$, as shown in figure 2.2. The change, $d\mathbf{A}$ in $\mathbf{A}$ under this rotation is a tiny vector from the tip of $\mathbf{A}$ to the tip of $\mathbf{A} + d\mathbf{A}$. The figure illustrates that $d\mathbf{A}$ is perpendicular to both $\mathbf{A}$ and $\hat{\mathbf{n}}$. Moreover, if $\mathbf{A}$ makes an angle $\theta$ with the axis $\hat{\mathbf{n}}$, then, in magnitude, $|d\mathbf{A}| = A \sin\theta \, d\phi$, so that as a vector equation,

$$d\mathbf{A} = \hat{\mathbf{n}} \times \mathbf{A} \, d\phi.$$

This has the right direction and magnitude.

If you perform a second infinitesimal rotation, then the change will be some new $d\mathbf{A}'$ say. The total change in $\mathbf{A}$ is then $d\mathbf{A} + d\mathbf{A}'$, but since addition of vectors

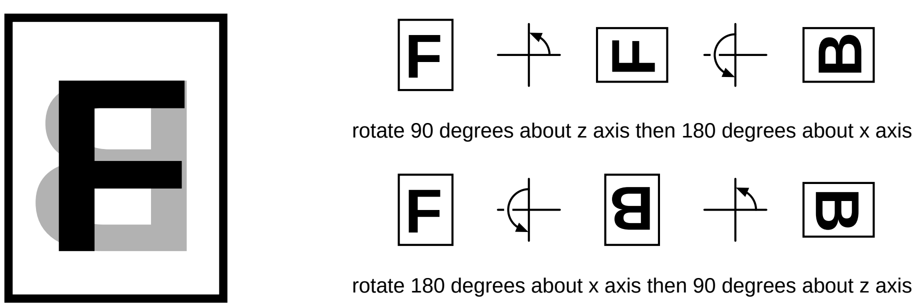


**Figure 2.1** Finite rotations do not commute. A sheet of paper has the letter "F" on the front and "B" on the back (shown light grey in the figure). Doing two finite rotations in different orders produces a different final result.

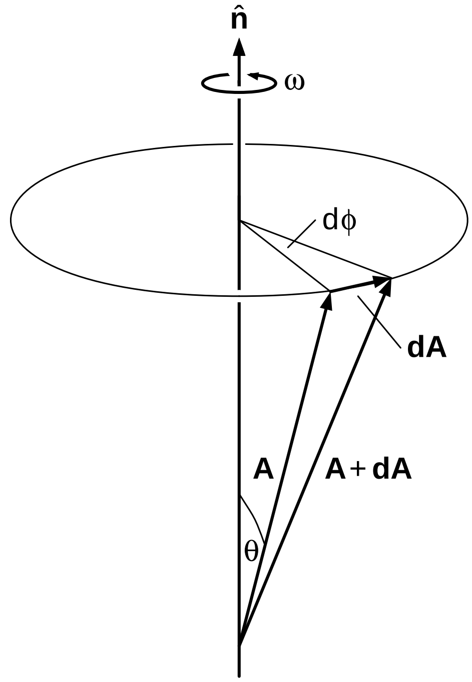


**Figure 2.2** A vector is rotated through an infinitesimal angle about an axis.

commutes, this is the same as $d\mathbf{A}' + d\mathbf{A}$. So, infinitesimal rotations *do* combine as vectors.

Now think of $\mathbf{A}$ as denoting a position vector, rotating around the axis with angular velocity $d\phi/dt = \dot{\phi}$, with the length of $\mathbf{A}$ fixed. This describes a particle rotating in a circle about the axis. The velocity of the particle is,

$$\mathbf{v} = \frac{d\mathbf{A}}{dt} = \hat{\mathbf{n}} \times \mathbf{A}\dot{\phi}.$$

We can define the vector *angular velocity*,

$$\boldsymbol{\omega} = \dot{\phi}\hat{\mathbf{n}},$$

and then,

$$\boxed{\frac{d\mathbf{A}}{dt} = \boldsymbol{\omega} \times \mathbf{A}}. \tag{2.1}$$

It's not necessary to think of $\mathbf{A}$ as a position vector, so this result describes the rate of change of any rotating vector of fixed length.

## 2.2 Moment of Inertia

We will consider the rotational motion of *rigid bodies*, where the relative positions of *all* the particles in the system are fixed. Specifying how one point in the body moves around an axis is then sufficient to specify how the whole body moves. The idea of a rigid body is clearly an idealisation. Real bodies are not rigid and will deform, however slightly, when subject to loads. Their constituents are also subject to random thermal motion. Nonetheless there are many situations where the deformation and any thermal motion can be ignored.

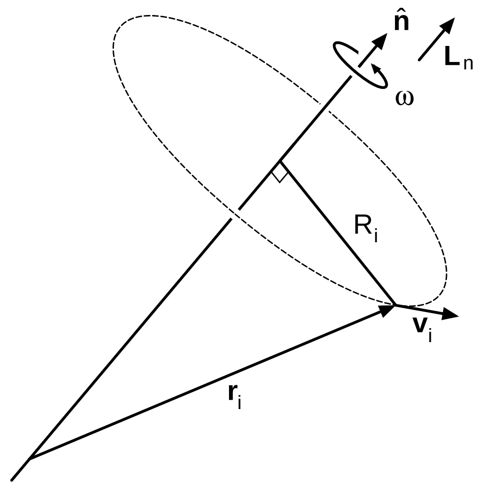


**Figure 2.3** Rigid body rotation about a fixed axis.

The general motion of a rigid body with a moving rotation axis is complicated, so we will specialise to a *fixed* axis at first. We can extend our analysis to *laminar* motion, where the axis can move, without changing its direction: an example is given by a cylinder rolling in a straight line down an inclined plane. We will later discuss precession, where the axis itself rotates.

For a rigid body rotating about a fixed axis, what property controls the angular acceleration produced by an external torque? The property will be the rotational analogue of mass (which tells you the linear acceleration produced by a given force). It is known as the *moment of inertia*, sometimes abbreviated (in these notes anyway) as *MoI*.

To find out how to define the MoI, look at the kinetic energy of rotation. Let $\boldsymbol{\omega} = \omega\hat{\mathbf{n}}$, so that $\hat{\mathbf{n}}$ specifies the rotation axis. Let $m_i$ be the mass of the $i$th particle in the body and let $R_i$ be the perpendicular distance of the $i$th particle from the rotation axis. The geometry is illustrated in figure 2.3. Since the body is rigid, $R_i$ is a fixed distance for each $i$ and $\omega$ is the same for all particles in the body. The kinetic energy is

$$T = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i R_i^2 \omega^2 = \frac{1}{2} I \omega^2,$$

where the last equality allows us to define the MoI about the given axis, according to,

$$\boxed{I \equiv \sum_i m_i R_i^2}.$$

The contribution of an element of mass to $I$ grows quadratically with its distance from the rotation axis. Note the analogy between $\frac{1}{2}mv^2$ for the kinetic energy of a particle moving with speed $v$ and $\frac{1}{2}I\omega^2$ for the kinetic energy of a body with moment of inertia $I$ rotating with angular speed $\omega$.

If the position vector $\mathbf{r}_i$ of the $i$th particle is measured from a point on the rotation axis, then $\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i$ and $v_i = |\boldsymbol{\omega} \times \mathbf{r}_i| = R_i\omega$. This is an application of the result in equation (2.1) for the rate of change of a rotating vector.

The moment of inertia is one measure of the mass distribution of an object. Other characteristics of the mass distribution we have already met are the total mass and the location of the centre of mass.

For a continuous mass distribution, simply replace the sums over discrete particles with integrals over the mass distribution,

$$\boxed{I = \int_{\text{body}} R^2\, dm = \int_{\text{body}} R^2 \rho\, d^3\mathbf{r}}.$$

Here, $dm = \rho\, d^3\mathbf{r}$ is a mass element, $\rho$ is the mass density and $d^3\mathbf{r}$ is a volume element.

It is sometimes convenient to use the *radius of gyration*, $k$, defined by

$$\boxed{I \equiv Mk^2}.$$

A single particle of mass equal to the total mass of the body at distance $k$ from the rotation axis will have the same moment of inertia as the body.

Now look at the component $L_n$ in the direction of the rotation axis of the (vector) angular momentum about some point on the axis (see figure 2.3). This is obtained by summing all the contributions of momenta perpendicular to the axis times the perpendicular separation from the axis,

$$\boxed{L_n = \sum_i R_i (m_i R_i \omega) = I\omega}.$$

The subscript $n$ labels the rotation axis. Note that the angular momentum of the $i$th particle is $\mathbf{L}_i = \mathbf{r}_i \times m_i \mathbf{v}_i$, and the component of this in the direction of $\hat{\mathbf{n}}$ is,

$$\hat{\mathbf{n}} \cdot (\mathbf{r}_i \times m_i \mathbf{v}_i) = \hat{\mathbf{n}} \cdot (\mathbf{r}_i \times m_i (\boldsymbol{\omega} \times \mathbf{r}_i)) = m_i R_i^2 \omega,$$

which is just what appears in the sum giving $L_n$.

If $\hat{\mathbf{n}}$ is a symmetry axis then $L_n$ is the only non-zero component of the total angular momentum $\mathbf{L}$. However, in general, $\mathbf{L}$ need not lie along the axis, or equivalently, $\mathbf{L}$ *need not be parallel to* $\boldsymbol{\omega}$.

Taking components of the angular equation of motion, $\boldsymbol{\tau} = d\mathbf{L}/dt$ along the axis gives,

$$\tau_n = \frac{dL_n}{dt} = I\dot{\omega} = I\ddot{\phi},$$

if $\phi$ measures the angle through which the body has rotated from some reference position.

## 2.3 Two Theorems on Moments of Inertia

### 2.3.1 Parallel Axis Theorem

$$\begin{aligned} I_{\text{CM}} &= \text{Moment of Inertia (MoI) about axis through centre of mass (CM)} \\ I &= \text{MoI about parallel axis at distance } d \text{ from axis through CM} \end{aligned}$$

The parallel axis theorem states:

$$\boxed{I = I_{\text{CM}} + Md^2},$$

where $M$ is the total mass. To prove this result, choose coordinates with the $z$-axis along the direction of the two parallel axes, as shown in figure 2.4. Then,

$$I = \sum_{i=1}^{N} m_i (x_i^2 + y_i^2).$$

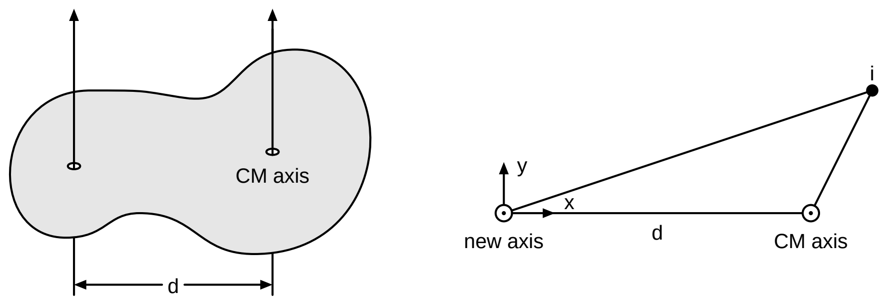


**Figure 2.4** Parallel axis theorem. In the right hand figure, we are looking vertically down in the $z$ direction.

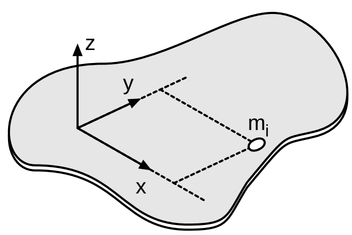


**Figure 2.5** Perpendicular axis theorem for thin flat plates.

We can also choose the $x$-direction to run from the new axis to the CM axis. Then,

$$x_i = d + \rho_{ix} \quad \text{and} \quad y_i = \rho_{iy}$$

where $\rho_{ix}$ and $\rho_{iy}$ are coordinates with respect to the CM. The expression for $I$ becomes:

$$I = \sum_{i=1}^{N} m_i\left((d+\rho_{ix})^2 + \rho_{iy}^2\right) = \sum_{i=1}^{N} m_i\left(\rho_{ix}^2 + \rho_{iy}^2 + d^2 + 2d\rho_{ix}\right).$$

The last term above contains $\sum m_i \rho_{ix}$ which vanishes by the definition of the CM. The remaining terms give $I_{\mathrm{CM}}$ and $Md^2$ and the result is proved.

### 2.3.2 Perpendicular Axis Theorem

This applies for thin flat plates of arbitrary shapes, which we take to lie in the $x$-$y$ plane, as shown in figure 2.5. Let $I_x$, $I_y$ and $I_z$ be the MoI about the $x$, $y$ and $z$ axes respectively. The perpendicular axis theorem states:

$$\boxed{I_z = I_x + I_y}.$$

The proof of this is very quick. Just observe that since we have a thin flat plate, then

$$I_x = \sum_{i=1}^{N} m_i y_i^2 \quad \text{and} \quad I_y = \sum_{i=1}^{N} m_i x_i^2.$$

But

$$I_z = \sum_{i=1}^{N} m_i (x_i^2 + y_i^2),$$

and the result is immediate.

In both these results we have assumed discrete distributions of point masses. For continuous mass distributions, simply replace the sums by integrations. For example,

$$I_z = \sum_{i=1}^{N} m_i (x_i^2 + y_i^2) \longrightarrow \int (x^2 + y^2)\, dm.$$

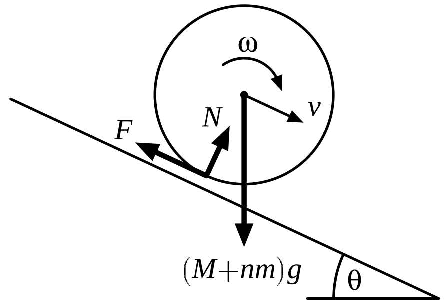


**Figure 2.6** Wheel rolling down a slope.

## 2.4 Examples

**Moment of Inertia of a Thin Rod** Find the moment of inertia of a uniform thin rod of length $2a$ about an axis perpendicular to the rod through its centre of mass. Also find the moment of inertia about a parallel axis through the end of the rod.

Let $\rho$ be the mass per unit length of the rod and let $x$ measure position along the rod starting from the centre of mass (so $-a \leq x \leq a$). For an element of the rod of length $dx$ the mass is $\rho dx$ and the moment of inertia of the element is $\rho x^2\, dx$. Therefore the total moment of inertia is given by the integral:

$$I_{\rm CM} = \int_{-a}^{a} \rho x^2\, dx = \frac{2}{3}\rho a^3.$$

The total mass is $m = 2\rho a$, and therefore,

$$I_{\rm CM} = \frac{1}{3}ma^2.$$

Applying the parallel axis theorem, the moment of inertia about one end of the rod is,

$$I_{\rm end} = I_{\rm CM} + ma^2 = \frac{4}{3}ma^2.$$

**Spoked Wheel** A wheel of radius $a$ comprises a thin rim of mass $M$ and $n$ spokes, each of mass $m$, which may be considered as thin rods terminating at the centre of the wheel. If the wheel rolls without slipping down a plane inclined at angle $\theta$ to the horizontal, as depicted in figure 2.6, what is the linear acceleration of its centre of mass?

We will apply the angular equation of motion about the centre of mass (see equation (1.7) on page 7), and the linear equation of motion (see equation (1.2) on page 2) in a direction parallel to the sloping plane. If the angular velocity of the wheel is $\omega$, then the no-slip condition says that its speed is $v = a\omega$. Choose directions so that $\omega$ and $v$ are both positive when the wheel rolls downhill.

The angular equation of motion applied to the wheel about its centre of mass says $\tau_{\rm CM}^{\rm ext} = I_{\rm CM}\,\dot{\omega}$. The external torque comes from the frictional force $F$ acting up the sloping plane at the point of contact with the wheel. Using the result above for the MoI of a rod (remembering that the rod length is now $a$ instead of $2a$), we find,

$$I_{\rm CM} = Ma^2 + \frac{n}{3}ma^2.$$

The angular equation of motion then gives,

$$Fa = (Ma^2 + \frac{n}{3}ma^2)\dot{\omega}.$$

The component of the linear equation of motion in a direction down the plane gives,

$$-F + (M+nm)g\sin\theta = (M+nm)a\dot{\omega}.$$

We now eliminate $F$ and solve for $a\dot{\omega}$, which gives the linear acceleration as,

$$a\dot{\omega} = \frac{3(M+nm)g\sin\theta}{6M+4nm}.$$

Alternatively, since the normal reaction ($N$ in figure 2.6) and frictional forces on the wheel do no work, we can apply the conservation of the kinetic plus (gravitational) potential energy. Applying our result in equation (1.4) on page 3 for the kinetic energy of a system, we find:

$$\frac{1}{2}(M+nm)v^2 + \frac{1}{2}I_{\mathrm{CM}}\omega^2 - (M+nm)gx\sin\theta = \text{const},$$

where $x$ is the distance moved starting from some reference point. Using $v = \dot{x} = a\omega$ and differentiating with respect to time gives

$$\frac{1}{3}(6M+4nm)\dot{x}\ddot{x} = (M+nm)g\sin\theta\,\dot{x},$$

which leads to the same result as before for the acceleration $a\dot{\omega} = \ddot{x}$.

## 2.5 Precession

Spinning bodies tend to *precess* under the action of a gravitational torque. We'll work out the steady precession rate for a spinning top. Figure 2.7 shows a top supported at a fixed pivot point. We will apply the angular equation of motion $\boldsymbol{\tau} = d\mathbf{L}/dt$ about the pivot. As drawn, the torque about the pivot due to the weight of the top points into the paper. Hence, the angular momentum $\mathbf{L}$ of the top must change by moving into the paper. If the top is spinning very fast about its axis, then $\mathbf{L}$ is, to a very good approximation, aligned with the top's axis. So, the top will tend to turn bodily, or *precess* around a vertical axis. It may help to think of the torque $\boldsymbol{\tau}$ pushing the tip of $\mathbf{L}$ around.

We can calculate the precession frequency quite easily. Assume that $\mathbf{L}$ is large so that the total angular momentum of the top is given entirely by the spin, and ignore any contribution due to the slow precession of the top about the vertical axis. The torque is given by,

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F},$$

where $\mathbf{r}$ is the vector from the pivot to the top's centre of mass and $\mathbf{F} = m\mathbf{g}$ is the top's weight. In magnitude,

$$\tau = mgr\sin\alpha,$$

where the top's axis makes an angle $\alpha$ with the vertical.

If the top precesses through an infinitesimal angle $d\phi$ about the vertical axis, then the magnitude of the change in $\mathbf{L}$ is,

$$dL = L\,d\phi\sin\alpha.$$

If $\dot{\phi} = \omega_{\mathrm{p}}$ is the precession angular velocity, then,

$$\frac{dL}{dt} = L\omega_{\mathrm{p}}\sin\alpha.$$

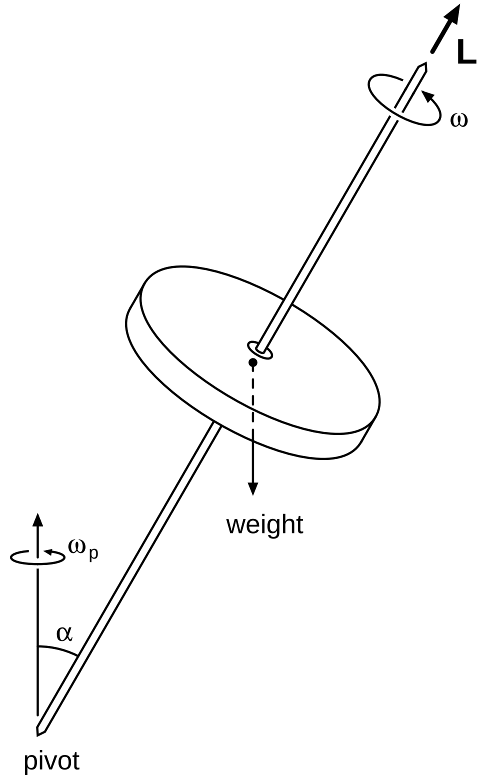


**Figure 2.7** A spinning top will precess under gravity.

Applying the equation of motion, taking the magnitude of both sides, gives:

$$mgr\sin\alpha = L\omega_{\mathrm{p}}\sin\alpha.$$

The $\sin\alpha$ terms cancel and the final answer comes out independent of the angle which the top makes with the vertical. The precession angular velocity is given by,

$$\boxed{\omega_{\mathrm{p}} = \frac{mgr}{L}}.$$

A full treatment of the motion of a top is complicated. Steady precession is a special motion: in general the top tends to nod up and down, or nutate, as it precesses.

## 2.6 Gyroscopic Navigation

A gyrocompass is a spinning top mounted in a frame so that its axis is constrained to be horizontal with respect to the Earth, see figure 2.8. As the Earth turns, the axis turns with it, causing the end of the axis labelled $A$ in the figure to be raised upwards and the end $B$ to be pushed down (as seen from a fixed frame not attached to the Earth). This means that there is a torque on the gyroscope which is perpendicular to the spin angular momentum $\mathbf{L}$ and points between the North and West when the compass is oriented as in the figure.

From the angular equation of motion, $\boldsymbol{\tau} = d\mathbf{L}/dt$, this torque will tend to push $\mathbf{L}$ towards the North. If $\mathbf{L}$ points between North and West, the torque again tries to line up $\mathbf{L}$ with the North-South axis. The gyrocompass will thus tend to oscillate with

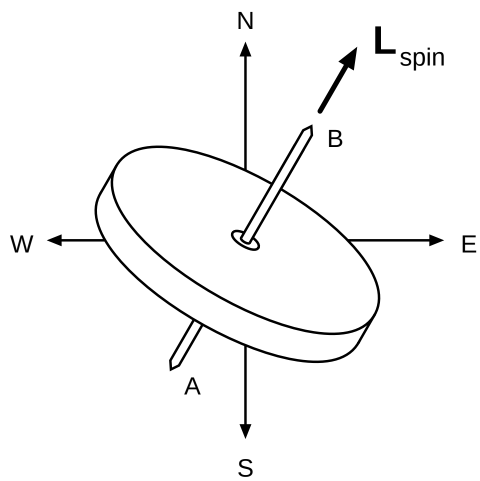


**Figure 2.8** A gyrocompass.

its spin direction oscillating about the N-S axis. If you apply some damping, then it will tend to settle down with its spin along the N-S line.

## 2.7 Inertia Tensor*

Now let's look at the moment of inertia in more detail. So far when we've considered the MoI for a body rotating around a fixed axis, we've always looked at the component $L_n$ of the angular momentum $\mathbf{L}$ along the direction of the axis $\hat{\mathbf{n}}$. Now let's look at *all* the components of $\mathbf{L}$. From the definition of angular momentum we have,

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i = \sum_{i=1}^{N} \mathbf{r}_i \times m_i(\boldsymbol{\omega} \times \mathbf{r}_i) = \sum_{i=1}^{N} m_i(\mathbf{r}_i \cdot \mathbf{r}_i \, \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \mathbf{r}_i \, \mathbf{r}_i),$$

where we have used $\mathbf{p}_i = m_i \mathbf{v}_i = m_i \boldsymbol{\omega} \times \mathbf{r}_i$ and $\boldsymbol{\omega} = \omega \hat{\mathbf{n}}$. We also applied a standard result for the vector triple product, $\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) = \mathbf{r}_i \cdot \mathbf{r}_i \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \mathbf{r}_i \mathbf{r}_i$. Rewrite this as a matrix equation giving the components of $\mathbf{L}$ in terms of the components of $\boldsymbol{\omega}$ (the summations run over $i = 1, \ldots, N$):

$$\begin{aligned}
\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} &= \begin{pmatrix} \sum m_i(y_i^2 + z_i^2) & -\sum m_i x_i y_i & -\sum m_i x_i z_i \\ -\sum m_i y_i x_i & \sum m_i(z_i^2 + x_i^2) & -\sum m_i y_i z_i \\ -\sum m_i z_i x_i & -\sum m_i z_i y_i & \sum m_i(x_i^2 + y_i^2) \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \\
&= \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}.
\end{aligned}$$

This is given more succinctly as,

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega},$$

where $\mathbf{I}$ is the matrix, known as the *inertia tensor* which acts on $\boldsymbol{\omega}$ to give $\mathbf{L}$. Remembering that $\boldsymbol{\omega} = \omega \hat{\mathbf{n}}$, our old results are recovered from,

$$T = \frac{1}{2} \hat{\mathbf{n}}^T \mathbf{I} \hat{\mathbf{n}} \omega^2 \qquad \text{and} \qquad L_n = \hat{\mathbf{n}}^T \mathbf{I} \hat{\mathbf{n}} \omega,$$

so we can define

$$I_n \equiv \hat{\mathbf{n}}^T \mathbf{I}\, \hat{\mathbf{n}}$$

as the moment of inertia about the axis $\hat{\mathbf{n}}$. This corresponds to what we called $I$ earlier, when we didn't make explicit reference to the rotation axis we were using. Here we are thinking of a matrix notation, so $\hat{\mathbf{n}}^T$ means the transpose of $\hat{\mathbf{n}}$, which gives a row vector.

The result $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$ shows quite clearly that although the angular momentum depends linearly on $\boldsymbol{\omega}$ it does *not* have to be parallel to $\boldsymbol{\omega}$. One important place where this matters is wheel balancing on cars. A wheel is unbalanced precisely when $\mathbf{L}$ and $\boldsymbol{\omega}$ are not parallel. Then, as the wheel rotates with $\boldsymbol{\omega}$ fixed, $\mathbf{L}$ describes a cone so $d\mathbf{L}/dt \neq 0$. Therefore a torque must be applied and you feel "wheel wobble." This is corrected by adding small masses to the wheel rim to adjust $\mathbf{I}$ to make $\mathbf{L}$ and $\boldsymbol{\omega}$ line up. In general, since $\mathbf{I}$ is a symmetric matrix, it can be diagonalised. This means it is always possible to choose a set of axes in the body for which $\mathbf{I}$ has non zero elements only along the diagonal. If you rotate the body around one of these *principal axes*, $\mathbf{L}$ and $\boldsymbol{\omega}$ will be parallel.

### 2.7.1 Free Rotation of a Rigid Body — Geometric Description*

Consider the rotational motion of a rigid body moving freely under no forces (or, a rigid body falling freely in a uniform gravitational field so that there are no torques about the CM; or, a rigid body freely pivoted at the CM).

If there are no torques acting, the total angular momentum, $\mathbf{L}$, must remain constant. It is convenient to choose axes fixed in the body, aligned with its principal axes of inertia. These body axes are themselves rotating, so in these coordinates the components of $\mathbf{L}$ along the axes may change (see chapter 4 on rotating coordinate systems). However, $|\mathbf{L}|$ is still fixed, so that $\mathbf{L}\cdot\mathbf{L} = L^2 = \text{const}$. Expressed in the body coordinates, this reads:

$$L^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2.$$

Furthermore, since there is no torque, the rotational kinetic energy is fixed, $T =$ const. Expressed in the body coordinates, this second conservation condition reads:

$$2T = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2.$$

The components of the angular velocity simultaneously satisfy two different equations. These equations specify two ellipsoids and $\boldsymbol{\omega}$ must lie on the line given by their intersection.

Suppose that all three principal moments of inertia are unequal, as is the case for, say, a book or a tennis racket. We'll take $I_1 < I_2 < I_3$. Now, start spinning the object with angular velocity of magnitude $\omega$ aligned along the $I_1$ axis. Angular momentum conservation says that the maximum magnitude of the component of $\boldsymbol{\omega}$ along the $I_2$ axis in the subsequent motion is $\omega I_1/I_2$, while kinetic energy conservation says the maximum magnitude of this component is $\omega\sqrt{I_1/I_2}$. Since $I_1 < I_2$, we find that the maximum component allowed by kinetic energy conservation is bigger, so that the kinetic energy ellipsoid lies *outside* the angular momentum ellipsoid along the $I_2$ axis. Likewise, since $I_1 < I_3$, the kinetic energy ellipsoid lies *outside* the angular momentum ellipsoid in the $I_3$ direction. Therefore, the intersection of the two ellipsoids comprises just two points, along the positive and negative $I_1$ directions. This is enough to tell you that rotation about the $I_1$ axis is stable — see figure 2.9(a).

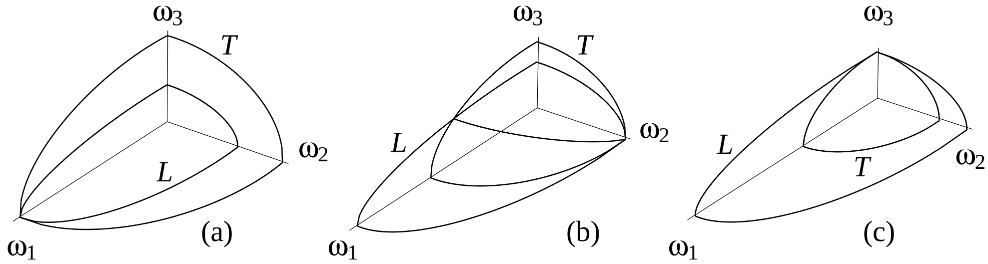


**Figure 2.9** Free rotation of a rigid body. The diagrams show the (first octants of the) kinetic energy and angular momentum ellipsoids for the free rotation of a rigid body with all three principal moments of inertia different, $I_1 < I_2 < I_3$. In (a) the rotation is stable with $\boldsymbol{\omega}$ pointing along the $I_1$ direction. In (b) the two ellipsoids intersect in a line, showing that rotation about the $I_2$ axis is unstable. In (c) the rotation is stable with $\boldsymbol{\omega}$ pointing in the $I_3$ direction.

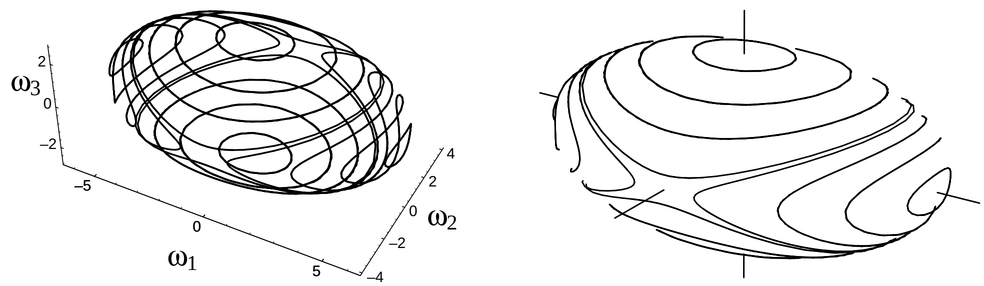


**Figure 2.10** Curves showing the time variation of angular velocity for a freely rotating object. The curves all lie on the ellipsoid of constant kinetic energy, and each one is given by the intersection of this ellipsoid with a similar ellipsoid of constant (magnitude of) angular momentum. On the left the full curves are shown, while on the right, parts of the curves on the “back” of the kinetic energy ellipsoid are hidden. The closed loops around the $I_1$ and $I_3$ axes show that the rotation is stable about these two axes.

A similar argument holds if you start with the angular velocity lined up along the $I_3$ axis, although in this case the angular momentum ellipsoid lies outside the kinetic energy ellipsoid, with the intersection only at two points along the positive and negative $I_3$ axes. Thus, rotation about the axis with the largest moment of inertia is also stable — see figure 2.9(c).

The final case we consider is where the initial angular velocity is aligned along the $I_2$ axis. Now, since $I_2 > I_1$, the angular momentum ellipsoid lies outside the kinetic energy ellipsoid in the $I_1$ direction, but, since $I_2 < I_3$, the angular momentum ellipsoid lies inside the kinetic energy ellipsoid in the $I_3$ direction. This means that there is a whole line of points where the two ellipsoids intersect — see figure 2.9(b). In turn, this tells you that rotation about the axis with intermediate moment of inertia is unstable: any small misalignment can be amplified and the object will be observed to “tumble” as it spins. It is easy to demonstrate this for yourself by throwing a book in the air, spinning it about each of its three principal axes in turn.

These three cases are illustrated in figure 2.9. Figure 2.10 shows the time variation of $\omega$ for the freely rotating body: each continuous curve shows the time variation of the components of $\omega$. The curves all lie on the surface of the ellipsoid of constant kinetic energy, and each curve is given by the intersection of this ellipsoid with an ellipsoid of constant angular momentum.