5

Simple Harmonic Motion*

Note: this section is not part of the syllabus for PHYS2006. You should already be familiar with simple harmonic motion from your first year course PHYS1015 *Motion and Relativity*. This section is included for completeness and as a reminder.

5.1 Simple Harmonic Motion

This is one of the most important phenomena in physics: it applies to the description of small oscillations of any system about a position of stable equilibrium.

Work in one dimension, so that one coordinate describes the position of the system (e.g. the displacement from the equilibrium position of a spring, the angle of a pendulum from the vertical). Only conservative forces do work, so there is a potential $V(x)$. Choose coordinates so that $x=0$ is a position of stable equilibrium. This means

$$F(x{=}0)=0, \qquad -\left.\frac{dV}{dx}\right|_0 = 0.$$

As long as x remains small, we can expand the potential:

$$V(x) = V(0) + xV'(0) + \frac{1}{2}x^2V''(0) + \cdots$$

However, $V'(0)=0$, since $x=0$ is a position of equilibrium, so the first derivative term vanishes. Letting $k=V''(0)$ (k is just the force constant for a spring force) and choosing our zero of potential energy so that $V(0)=0$, we find:

$$V(x) = \frac{1}{2}kx^2 + \cdots.$$

The corresponding force is $F(x)=-kx$. We ignore the special case $k=0$, when the expansion of V begins at higher order. If $k<0$ then the equilibrium is unstable, and the system will move out of the region where our approximation is valid. Hence we will look at displacements around positions of stable equilibrium for which $k>0$.

We define a *Simple Harmonic Oscillator* as a one-dimensional problem with:

$$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega_0^2x^2$$

where $k>0$ and we have defined $\omega_0^2 = k/m$.

A mass oscillating on a Hooke's law spring is a simple harmonic oscillator. Small oscillations of a simple pendulum are simple harmonic.

5.1.1 General Solution

The equation of motion for the simple harmonic oscillator is

$$\ddot{x} + \omega_0^2 x = 0.$$

This is a second order homogeneous linear differential equation, meaning that the highest derivative appearing is a second order one, each term on the left contains exactly one power of x, $\dot{x}$ or $\ddot{x}$ (there is no $\dot{x}$ term in this case) and there is no term (a constant or a function of time) on the right.

Two independent solutions of this are $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$. The general solution is a linear combination of these, which can be written in several forms:

$$\begin{aligned} x &= A\cos(\omega_0 t) + B\sin(\omega_0 t) \\ &= C\cos(\omega_0 t + \delta) \\ &= D\sin(\omega_0 t + \varepsilon) \\ &= \mathrm{Re}\left(\alpha e^{i\omega_0 t}\right) \\ &= \mathrm{Im}\left(\beta e^{-i\omega_0 t}\right) \end{aligned}$$

where A, B, C, D, δ and ε are real constants, and α and β are complex constants. Use whichever solution is most convenient. We will often use the complex exponential forms, so we will need to remember that the physical solutions are found by taking the real or imaginary parts. Some terminology associated with the simple harmonic oscillator is:

angular frequency	ω_0
period	$T = 2\pi/\omega_0$
amplitude	$a = \lvert C\rvert = \lvert D\rvert = \sqrt{A^2 + B^2} = \lvert\alpha\rvert = \lvert\beta\rvert$

The arguments of the sine or cosine in $\cos(\omega_0 t + \delta)$ and $\sin(\omega_0 t + \varepsilon)$ are called the phase. The period of a simple harmonic oscillator is independent of the amplitude: this is a special property, not true for oscillators in general.

5.2 Damped Harmonic Motion

We'll assume that a damping force proportional to speed is present,

$$F_{\text{damping}} = -2m\gamma\dot{x}.$$

This equation defines γ (note that in defining γ we have pulled out one factor of m for convenience: γ could still itself depend on m). *Warning*: many authors use $\gamma/2$ in place of γ.

In general, the damping can be some power series in $\dot{x}$. We approximate by keeping the linear term only. In practice, this turns out to work well: the viscously damped harmonic oscillator is a very useful model for all sorts of physical systems.

The equation of motion has become:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0.$$

This is still a linear, homogeneous second order differential equation. We try a solution of the form

$$x = Ae^{\Omega t}$$

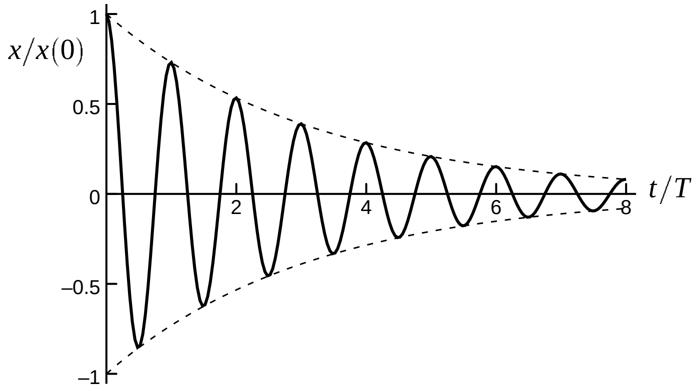

Figure 5.1 Amplitude as a function of time for a lightly damped harmonic oscillator. The time is measured in units of the "period" $T = 2\pi/\omega$. The dashed lines show the exponentially damped envelope of the oscillatory motion.

where A and Ω may be complex (and we take the real part at the end). We will do this same trick of using a complex exponential many times more. Substituting our trial solution gives:

$$(\Omega^2 + 2\gamma\Omega + \omega_0^2)Ae^{\Omega t} = 0.$$

Since the equation is linear, A is arbitrary, and we want it non zero in order to have a non-trivial solution. The factor in brackets then gives a quadratic for Ω: the two roots of this will provide us with our two independent solutions.

5.2.1 Small Damping: $\gamma^2 < \omega_0^2$

The roots of the quadratic are

$$\Omega = -\gamma \pm i\omega, \qquad \text{where} \qquad \omega = \sqrt{\omega_0^2 - \gamma^2}.$$

A solution may be written $x = \mathrm{Re}\,(A_1 e^{i\omega t} + A_2 e^{-i\omega t})e^{-\gamma t}$, which can be reexpressed as:

$$x = Be^{-\gamma t}\cos(\omega t + \delta).$$

This describes an oscillation with "frequency" $\omega = \sqrt{\omega_0^2 - \gamma^2}$ and exponentially decaying "amplitude" $Ae^{-\gamma t}$, as illustrated in figure 5.1. The quotes are here because the motion is no longer periodic, so there is not really a frequency. However, you could use the time between the system crossing $x = 0$ in the *same* direction as a measure of a "period", since this time is $2\pi/\omega$. If the damping is truly small, then the oscillations will appear to have amplitude $Ae^{-\gamma t}$ if you watch them for a short interval around time t.

In one "period", $T = 2\pi/\omega$ of a lightly damped oscillator's motion, the fractional energy loss is found by comparing the total energy at the start of the period and at the end. For any time, t, the fractional loss is given by

$$\frac{\Delta E}{E} = \frac{E(t) - E(t+T)}{E(t)} = 1 - e^{-2\gamma T}.$$

When the damping is very small, $\gamma/\omega_0 \ll 1$, we have $\omega \approx \omega_0$ and then

$$\frac{\Delta E}{E} \approx 2\pi\frac{2\gamma}{\omega_0} \equiv \frac{2\pi}{Q},$$

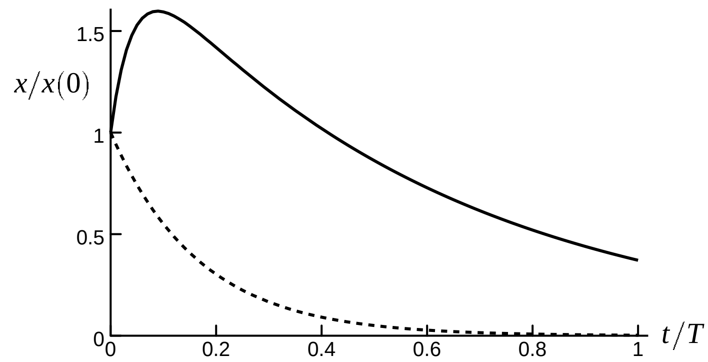

Figure 5.2 Amplitude as a function of time for heavily damped (solid curve) and critically damped (dashed curve) harmonic oscillators. The time is measured in units of the natural period $T = 2\pi/\omega_0$ of the oscillator when the damping is switched off.

which defines the *quality factor* Q. *Warning:* definitions of Q vary from author to author.

5.2.2 Large Damping: $\gamma^2 > \omega_0^2$

The roots of the quadratic are

$$\Omega = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

and a solution may be written

$$x = Ae^{-(\gamma+\sqrt{\gamma^2-\omega_0^2})t} + Be^{-(\gamma-\sqrt{\gamma^2-\omega_0^2})t}.$$

This is a sum of two exponentials, both decaying with time, illustrated by the solid curve in figure 5.2. The "B" exponential falls more slowly, so it dominates at large times. This case is sometimes referred to as "overdamped".

5.2.3 Critical Damping: $\gamma^2 = \omega_0^2$

In this special case the solutions for Ω are degenerate (the roots of the quadratic coincide). It looks as though there is just one solution. However, a second order differential equation *must* have two independent solutions. You can check by differentiating that the second solution in this case is

$$x = Bte^{-\gamma t},$$

so that the general solution becomes:

$$x = (A + Bt)e^{-\gamma t}.$$

The critically damped solution is illustrated by the dashed curve in figure 5.2. Critical damping is important: for example a measuring instrument should be critically damped so that the reading settles down as fast as possible without the response time being too slow.

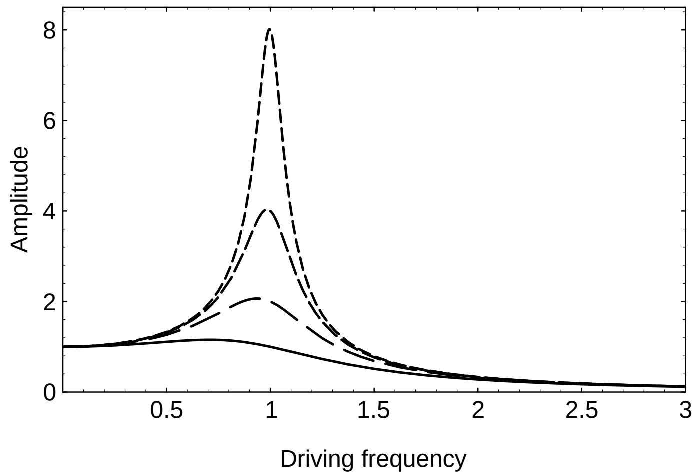

Figure 5.3 Amplitude of forced harmonic oscillator as a function of driving frequency (in units of natural frequency)

5.3 Driven damped harmonic oscillator

The equation of motion for a damped harmonic oscillator driven by an external force $F(t)$ is

$$m\ddot{x} + 2m\gamma\dot{x} + m\omega_0^2 x = F(t).$$

Consider the case of a periodic driving force,

$$F(t) = mf\cos(\omega t) = mf\mathrm{Re}\,(e^{i\omega t}),$$

and look for the *steady state* solution, when any *transient* damped solution has died away (the transients are solutions of the differential equation without the driving term $F(t)$, that is, a free damped oscillator). Look for a complex z which solves

$$\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = fe^{i\omega t},$$

and take the real part of z at the end. Try a trial solution $z = Ae^{i\omega t}$: the idea is that after a long time we expect the system to be oscillating with the same frequency as the driving force. More technically, the full solution of the differential equation is the sum of the solution we are about to find plus *any* solution of the undriven equation (without the $fe^{i\omega t}$ term). Because of the damping, the solution in the undriven case decays exponentially with time: we are interested in what happens after a long time when this *transient* solution has died out.

Returning to our trial solution, $z = Ae^{i\omega t}$ solves the equation if

$$(-\omega^2 + 2i\gamma\omega + \omega_0^2)Ae^{i\omega t} = fe^{i\omega t}.$$

Cancelling the $e^{i\omega t}$ from both sides and solving for A gives

$$A = \frac{f}{-\omega^2 + 2i\gamma\omega + \omega_0^2}.$$

Writing $A = |A|e^{-i\delta}$, we find that the oscillation amplitude $|A|$ and phase lag δ are given by,

$$|A| = \frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}$$

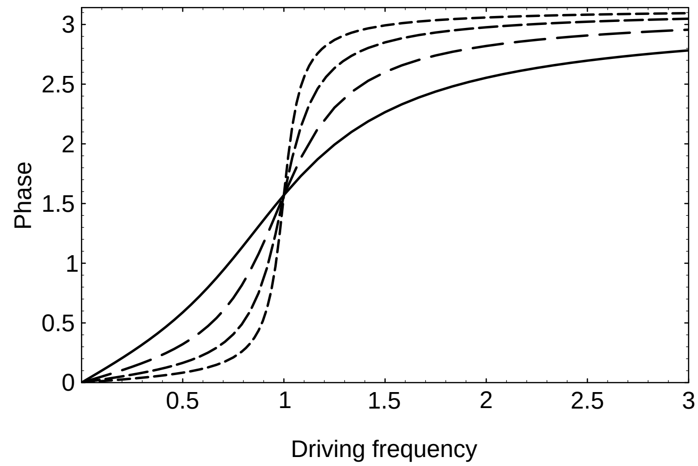

Figure 5.4 Phase lag of forced harmonic oscillator as a function of driving frequency (in units of natural frequency)

and

$$\tan\delta = \frac{2\gamma\omega}{\omega_0^2 - \omega^2}.$$

In figures 5.3 and 5.4 we plot the amplitude (actually $\omega_0^2|A|/f$) and the phase δ as functions of ω/ω_0, for four different values of the *quality factor* $Q = \omega_0/2\gamma$. The quality factor tells you about the ratio of the energy stored in the oscillator to the energy loss per cycle. As you move from solid to finer and finer dashed lines the Q values are 1, 2, 4 and 8 respectively.