6

Coupled Oscillators

In what follows, I will assume you are familiar with the simple harmonic oscillator and, in particular, the complex exponential method for finding solutions of the oscillator equation of motion. If necessary, consult the revision section on Simple Harmonic Motion in chapter 5.

6.1 Time Translation Invariance

Before looking at coupled oscillators, I want to remind you how time translation invariance leads us to use (complex) exponential time dependence in our trial solutions. Later, we will see that spatial translation invariance leads to exponential forms for the spatial parts of our solutions as well.

To examine the implication of time translation invariance, it's enough to consider a single damped harmonic oscillator, with equation of motion,

$$m\ddot{x} = -2m\gamma\dot{x} - m\omega_0^2 x,$$

where the two terms on the right are the damping and restoring forces respectively. We can rearrange this to,

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0.$$

To solve this equation, we used an ansatz (or guess) of the form

$$x = Ae^{\Omega t},$$

where A and Ω are in general complex (to get a physical solution you can use the real or imaginary parts of a complex solution). The reason that we could guess such a solution lies in time translation invariance.

What this invariance means is that we don't care about the origin of time. It doesn't matter what our clock read when we started observing the system. In the differential equation, this property appears because the time dependence enters only through time derivatives, *not* through the value of time itself. In terms of a solution $x(t)$, this means that:

$$\boxed{\text{if } x(t) \text{ is a solution, then so is } x(t+c) \text{ for any constant } c.}$$

The simplest possibility is that $x(t+c)$ is proportional to $x(t)$, with some proportionality constant $f(c)$, depending on c,

$$x(t+c) = f(c)x(t).$$

We can solve this equation by a simple trick. We differentiate with respect to c and then set $c = 0$ to obtain

$$\dot{x}(t) = \Omega x(t),$$

where Ω is just the value of $\dot{f}(0)$. The general solution of this linear first order differential equation is

$$x(t) = Ae^{\Omega t}.$$

We often talk about *complex* exponential forms because Ω must have a non-zero imaginary part if we want to get oscillatory solutions. In fact, from now on I will let $\Omega = i\omega$, so that ω is real for a purely oscillatory solution.

We can't just use *any* value we like for ω. The allowed values are determined by demanding that $Ae^{i\omega t}$ actually solves the equation of motion:

$$(-\omega^2 + 2i\gamma\omega + \omega_0^2)Ae^{i\omega t} = 0.$$

If we are to have a non-trivial solution, A should not vanish. The factor in parentheses must then vanish, giving a quadratic equation to determine ω. The two roots of the quadratic give us two independent solutions of the original second order differential equation.

6.2 Normal Modes

We want to generalise from a single oscillator to a set of oscillators which can affect each others' motion. That is to say, the oscillators are *coupled*.

If there are n oscillators with positions $x_i(t)$ for $i = 1, \ldots, n$, we will denote the "position" of the whole system by a vector $\mathbf{x}(t)$ of the individual locations:

$$\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t)).$$

The individual positions $x_i(t)$ might well be generalised coordinates rather than real physical positions.

The differential equations satisfied by the x_i will involve time dependence only through time derivatives, which means we can look for a time translation invariant solution, as described above. This means all the oscillators must have the same complex exponential time dependence, $e^{i\omega t}$, where ω is real for a purely oscillatory motion. The solution then takes the form,

$$\mathbf{x}(t) = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} e^{i\omega t},$$

where the A_i are *constants*. This describes a situation where all the oscillators move with the *same frequency*, but, in general, different phases and amplitudes: the oscillators' displacements are in fixed ratios determined by the A_i. This kind of motion is called a *normal mode*. The *overall* normalisation is arbitrary (by linearity of the differential equation), which is to say that you can multiply all the A_i by the same constant and still have the same normal mode.

Our job is to discover which ω are allowed, and then determine the set of A_i corresponding to each allowed ω. We will find precisely the right number of normal modes to provide all the independent solutions of the set of differential equations. For n oscillators obeying second order coupled equations there are $2n$ independent solutions: we will find n coupled normal modes which will give us $2n$ real solutions when we take the real and imaginary parts.

Once we have found all the normal modes, we can construct *any* possible motion of the system as a linear combination of the normal modes. Compare this with Fourier analysis, where any periodic function can be expanded as a series of sines and cosines.

6.3 Coupled Oscillators

Take a set of coupled oscillators described by a set of coordinates $q_1, \ldots, q_n$. In general the potential $V(q)$ will be a complicated function which couples all of these oscillators together. Consider *small* oscillations about a position of stable equilibrium, which (by redefining our coordinates if necessary) we can take to occur when $q_i = 0$ for $i = 1, \ldots, n$. Expanding the potential in a Taylor series about this point, we find,

$$V(q) = V(0) + \sum_i \left.\frac{\partial V}{\partial q_i}\right|_0 q_i + \frac{1}{2}\sum_{i,j} \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_0 q_i q_j + \cdots.$$

By adding an overall constant to V we can choose $V(0) = 0$. Since we are at a position of equilibrium, all the first derivative terms vanish. So the first terms that contribute are the second derivative ones. We define,

$$K_{ij} \equiv \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_0,$$

and drop all the remaining terms in the expansion. Note that K_{ij} is a constant symmetric (why?) $n \times n$ matrix. The corresponding force is thus

$$F_i = -\frac{\partial V}{\partial q_i} = -\sum_j K_{ij} q_j$$

and thus the equations of motion are

$$M_i \ddot{q}_i = -\sum_j K_{ij} q_j,$$

for $i = 1, \ldots, n$. Here the M_i are the masses of the oscillators, and K is a matrix of 'spring constants'. Indeed for a system of masses connected by springs, with each mass moving in the same single dimension, the coordinates can be taken as the real position coordinates, and then M is a (diagonal in this case) matrix of masses, while K is a matrix determined by the spring constants. Be aware however, that coupled oscillator equations occur more generally (for example in electrical circuits) where the q_is need not be actual coordinates but more general parameters describing the system (known as generalised coordinates) and in this case M and K play similar rôles even if they do not in actuality correspond to masses and spring constants.

To simplify the notation, we will write the equations of motion as a matrix equation. So we define,

$$\mathbf{M} = \begin{pmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_n \end{pmatrix}, \qquad \mathbf{K} = \begin{pmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{pmatrix}.$$

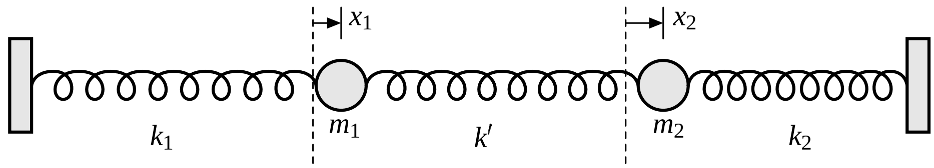

Figure 6.1 Two coupled harmonic oscillators. The vertical dashed lines mark the equilibrium positions of the two masses.

Likewise, let $\mathbf{q}$ and $\ddot{\mathbf{q}}$ be column vectors,

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}, \qquad \ddot{\mathbf{q}} = \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_n \end{pmatrix}.$$

With this notation, the equation of motion is,

$$\mathbf{M}\ddot{\mathbf{q}} = -\mathbf{K}\mathbf{q}, \qquad \text{or} \qquad \ddot{\mathbf{q}} = -\mathbf{M}^{-1}\mathbf{K}\mathbf{q},$$

where $\mathbf{M}^{-1}$ is the inverse of $\mathbf{M}$.

Now look for a normal mode solution, $\mathbf{q} = \mathbf{A}e^{i\omega t}$, where $\mathbf{A}$ is a column vector. We have $\ddot{\mathbf{q}} = -\omega^2\mathbf{q}$, and cancelling $e^{i\omega t}$ factors, gives finally,

$$\boxed{\mathbf{M}^{-1}\mathbf{K}\mathbf{A} = \omega^2\mathbf{A}}.$$

This is now an *eigenvalue equation*. The squares of the normal mode freqencies are the *eigenvalues* of $\mathbf{M}^{-1}\mathbf{K}$, with the column vectors $\mathbf{A}$ as the corresponding *eigenvectors*.

6.4 Example: Masses and Springs

As a simple example, let's look at the system shown in figure 6.1, comprising two masses m_1 and m_2 constrained to move along a straight line. The masses are joined by a spring with force constant k', and m_1 (m_2) is joined to a fixed wall by a spring with force constant k_1 (k_2). Assume that the equilibrium position of the system has each spring unstretched, and use the displacements x_1 and x_2 of the two masses away from their equilibrium positions as coordinates. The force on mass m_1 is then

$$F_1 = -k_1x_1 - k'(x_1 - x_2)$$

and on mass m_2

$$F_2 = -k_2x_2 - k'(x_2 - x_1).$$

(Note that these follow from a potential of form $V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k'(x_2 - x_1)^2 + \frac{1}{2}k_2x_2^2$.) You can check that Newton's 2nd law thus implies, in matrix form:

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1 + k' & -k' \\ -k' & k_2 + k' \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The eigenvalue equation we have to solve is:

$$\begin{pmatrix} (k_1 + k')/m_1 & -k'/m_1 \\ -k'/m_2 & (k_2 + k')/m_2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.$$

Now specialise to a case where $m_1 = m$, $m_2 = 2m$, $k_1 = k$, $k_2 = 2k$ and $k' = 2k$. The eigenvalue equation becomes,

$$\begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \frac{m}{k}\omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix},$$

or, setting $\lambda = m\omega^2/k$,

$$\begin{pmatrix} 3-\lambda & -2 \\ -1 & 2-\lambda \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For there to be a solution, the determinant of the 2×2 matrix in the last equation must vanish. This gives a quadratic equation for λ,

$$\lambda^2 - 5\lambda + 4 = 0,$$

with roots $\lambda = 1$ and $\lambda = 4$. The corresponding eigenfrequencies are $\omega = \sqrt{k/m}$ and $\omega = 2\sqrt{k/m}$. For each eigenvalue, there is a corresponding eigenvector. With $\lambda = 1$ you find $A_2 = A_1$, and with $\lambda = 4$ you find $A_2 = -A_1/2$. Note that just the ratio of the two A_i is determined: you can multiply all the A_i by a constant and stay in the same normal mode. This means that we are free to normalise the eigenvectors as we choose. It is common to make them have unit modulus, in which case the eigenfrequencies and eigenvectors are:

$$\begin{aligned} \omega &= \sqrt{\frac{k}{m}}, & \mathbf{A} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ \omega &= 2\sqrt{\frac{k}{m}}, & \mathbf{A} &= \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \end{aligned}$$

In the first normal mode, the two masses swing in phase with the same amplitude, and the middle spring remains unstretched. This could have been predicted: we have solved for a case where m_2 is twice the mass of m_1, and is attached to a wall by a spring with twice the force constant. Therefore, m_1 and m_2 would oscillate with the same frequency in the absence of the connecting spring.

In the second mode the two masses move out of phase with each other, and m_1 has twice the amplitude of m_2.

6.4.1 Weak Coupling and Beats

Now consider a case where the two masses are equal, $m_1 = m_2 = m$, and the two springs attaching the masses to the fixed walls are identical, $k_1 = k_2 = k$. From the symmetry of the setup, you expect one mode where the two masses swing in phase with the same amplitude, the central connecting spring remaining unstretched. In the second mode, the two masses again have the same amplitude, but swing out of phase, alternately approaching and receding from each other. This second mode will have a higher frequency (why?).

If the spring constant of the connecting spring is $k' = \varepsilon k$, you should check that applying the solution method worked through above gives the following eigenfrequencies and normal modes:

$$\begin{aligned} \omega_1 &= \sqrt{\frac{k}{m}}, & \mathbf{A}_1 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ \omega_2 &= \sqrt{(1+2\varepsilon)\frac{k}{m}}, & \mathbf{A}_2 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \end{aligned}$$

When the connecting spring has a very small force constant, $\varepsilon \ll 1$, so that the coupling is weak, the two normal modes have almost the same frequency. In this case it's possible to observe *beats* when a motion contains components from both normal modes. For example, suppose you start the system from rest by holding the left hand mass with a small displacement to the right, say d, keeping the right hand mass in its equilibrium position, and then letting go.

A general solution for the motion has the form,

$$\mathbf{x}(t) = c_1 \mathbf{A}_1 \cos(\omega_1 t) + c_2 \mathbf{A}_2 \cos(\omega_2 t) + c_3 \mathbf{A}_1 \sin(\omega_1 t) + c_4 \mathbf{A}_2 \sin(\omega_2 t).$$

Because the system starts from rest, you can immediately see (make sure you can!) that $c_3 = c_4 = 0$ in this case. Then the initial condition,

$$\mathbf{x}(0) = \begin{pmatrix} d \\ 0 \end{pmatrix},$$

gives,

$$\begin{pmatrix} d \\ 0 \end{pmatrix} = \frac{c_1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{c_2}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

which is solved by $c_1 = c_2 = d/\sqrt{2}$. So, the motion is given by:

$$\begin{aligned} x_1(t) &= \frac{d}{2}(\cos(\omega_1 t) + \cos(\omega_2 t)), \\ x_2(t) &= \frac{d}{2}(\cos(\omega_1 t) - \cos(\omega_2 t)). \end{aligned}$$

We can rewrite the sum and difference of cosines as products, leaving:

$$\begin{aligned} x_1(t) &= d \cos\left(\frac{\omega_2 - \omega_1}{2} t\right) \cos\left(\frac{\omega_1 + \omega_2}{2} t\right), \\ x_2(t) &= d \sin\left(\frac{\omega_2 - \omega_1}{2} t\right) \sin\left(\frac{\omega_1 + \omega_2}{2} t\right). \end{aligned}$$

Now you can see that each of x_1 and x_2 has a "fast" oscillation at the average frequency $(\omega_1 + \omega_2)/2$, modulated by a "slow" amplitude variation at the difference frequency $(\omega_2 - \omega_1)/2$. The displacements show the contributions of the two normal modes beating together, as illustrated in figure 6.2.

You can easily demonstrate beats by tying a length of cotton between two chairs and hanging two keys from it by further equal-length threads. Each key is a simple pendulum and the suspension thread provides a weak coupling between them. Start the system by pulling one of the keys to one side, with the other hanging vertically, and releasing, so that you start with one key swinging from side to side and the other at rest. The swinging key gradually reduces its amplitude, and at the same time the other key begins to move. Eventually, the first key will momementarily stop swinging, whilst the second key has reached full amplitude. The process then continues, and the swinging motion transfers back and forth between the two keys.

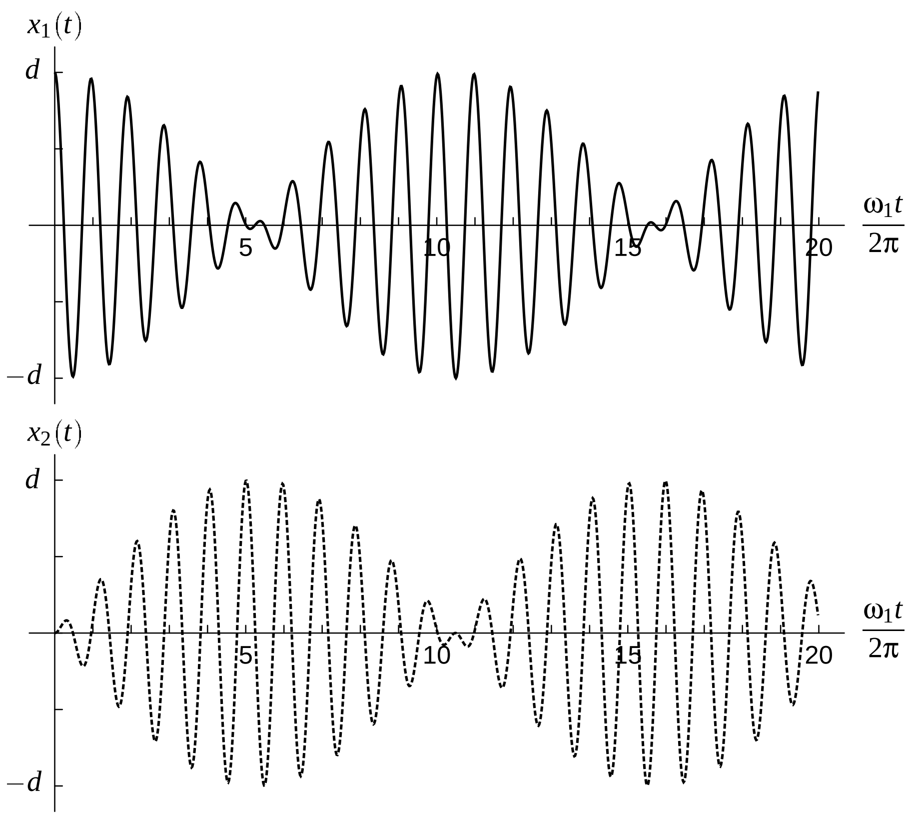

Figure 6.2 Displacements x_1 and x_2 as functions of time, starting with both masses at rest and $x_1(0) = d$, $x_2(0) = 0$. The displacement curve for x_2 is shown dashed. For this plot, the ratio ε of the spring force constants of the coupling (central) spring and either of the outer springs is 0.1. Time is plotted in units of the period of the lower frequency normal mode.