7

Normal Modes of a Beaded String

7.1 Equation of Motion

The system we will describe is a string stretched to tension T, carrying N beads, each of mass M, as shown in figure 7.1. The beads are equally spaced distance a apart, and the ends of the string are distance a from the first and last bead respectively. We will consider small transverse oscillations of the beads, with the ends of the string held in fixed positions.

If the displacement of the nth bead is u_n, we can work out its equation of motion by applying Newton's second law. Referring to the lower part of figure 7.1, we find:

$$M\ddot{u}_n = -T\,(\sin\psi + \sin\phi).$$

If the displacements are all small, then

$$\sin\psi \approx \frac{u_n - u_{n-1}}{a}, \qquad \text{and} \qquad \sin\phi \approx \frac{u_n - u_{n+1}}{a}.$$

Applying this approximation, the equations of motion are

$$\boxed{\ddot{u}_n = \frac{T}{Ma}(u_{n-1} - 2u_n + u_{n+1})}.$$

You get the same equation for longitudinal oscillations of a one-dimensional line of masses connected by identical springs, with C/M replacing T/Ma, where C is the spring constant of each spring.

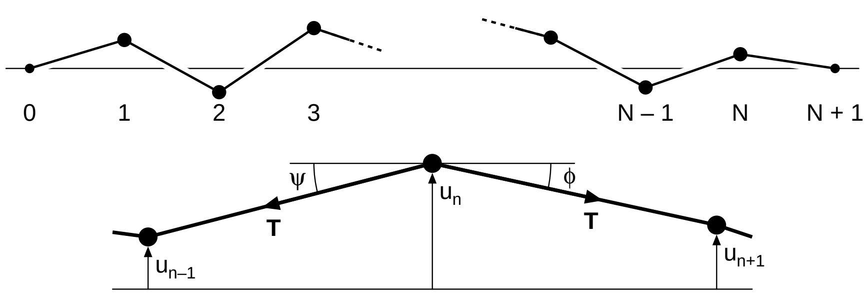

Figure 7.1 Transverse oscillations of a beaded string.

We can incorporate the boundary conditions, that the ends of the string are fixed, by requiring

$$u_0 = 0, \qquad u_{N+1} = 0.$$

You should convince yourself that these conditions give the right equations of motion for the first and Nth beads.

7.2 Normal Modes

We would like to find the normal modes of the beaded string. These are motions where all the beads oscillate with the same angular frequency ω:

$$u_n = A_n e^{i\omega t},$$

for some set of coefficients A_n. Substituting in the equation of motion gives,

$$\omega^2 A_n = \frac{T}{Ma}\left(-A_{n-1} + 2A_n - A_{n+1}\right). \tag{7.1}$$

This is a *recurrence relation* for the A_n — it is a discrete form of a differential equation. The boundary conditions are now incorporated as,

$$A_0 = A_{N+1} = 0.$$

We could solve for the A_n by viewing the recurrence relation as a matrix equation determining the column vector of the A_n's, like we did for systems with one or two degrees of freedom. Alternatively, we could apply known methods of solving recurrence relations. Rather than do either of these things, we will use some physical insight, allowing us almost to write down the solution with little effort. There are two key points:

- Suppose we actually had an *infinite* line of beads on a string. The infinite system has a *translation invariance*. If you jump one step (or any integer number of steps) left or right, the system looks the same. This will make it easy to find the normal modes of the infinite system.
- Each bead is connected to its two nearest neighbours only: the interaction is *local*. In the equation of motion, u_n is affected only by u_{n-1}, u_{n+1} and u_n itself, so the nth bead's displacement is affected *only* by the displacements of its two neighbours. Thus, if you can find a combination of normal modes of the infinite system which satisfies $A_0 = A_{N+1} = 0$, then you'll have found a mode of the finite system. You don't care what A_{-1}, A_{N+2} and so on are doing.

To repeat, we will look for normal modes by finding modes for an infinite line of beads and then selecting particular combinations of modes to satisfy the boundary conditions that the ends of the finite string are fixed.

7.2.1 Infinite System: Translation Invariance

Suppose we have already found a mode for the infinite string, with some set of displacement amplitudes A_n.

Now shift the system one step to the left. The translation invariance tells us it looks the same. This means that if the A_n gave us a mode with frequency ω, the shifted A'_n should give another mode with the *same* ω. That is, the new set of amplitudes,

$$A'_n = A_{n+1},$$

also give a mode.

Now let's look for a translation invariant mode, which reproduces itself when we do the shift. Since a mode is arbitrary up to an overall scale, this means,

$$A'_n = A_{n+1} = hA_n,$$

for some constant h, so that the new amplitudes are proportional to the old ones. Applying the last relation repeatedly shows that,

$$A_n = h^n A_0,$$

where A_0 is arbitrary and sets the overall scale. Given this set of A_n, we can find the corresponding angular frequency ω by substituting in the equation of motion in the form it appeared in equation (7.1). We find,

$$\omega^2 h^n A_0 = \frac{T}{Ma}\left(-h^{n-1}A_0 + 2h^n A_0 - h^{n+1}A_0\right).$$

Cancelling a common factor $h^n A_0$, leaves,

$$\omega^2 = \frac{T}{Ma}\left(2 - h - \frac{1}{h}\right). \tag{7.2}$$

This shows that h and $1/h$ give the same normal mode frequency. Conversely, if the frequency ω is fixed, the amplitudes A_n must be an arbitrary linear combination of the amplitudes for h and $1/h$. That is,

$$A_n = \alpha h^n + \beta h^{-n},$$

where α and β are constants.

We will find it convenient to set $h = e^{i\theta}$. The relation giving ω for a given h in equation (7.2) becomes a relation giving ω for a given θ according to,

$$\boxed{\omega^2 = 4\frac{T}{Ma}\sin^2(\theta/2)}. \tag{7.3}$$

The displacement of the nth bead is,

$$u_n = (\alpha e^{in\theta} + \beta e^{-in\theta})e^{i\omega t}. \tag{7.4}$$

7.2.2 Finite System: Boundary Conditions

The value of θ is fixed by the boundary conditions, and this in turn fixes ω. For the string of N beads with both ends fixed, we incorporate the boundary conditions by requiring

$$u_0 = 0, \qquad u_{N+1} = 0.$$

The $u_0 = 0$ condition requires that $\alpha = -\beta$, which makes u_n proportional to $\sin(n\theta)$ only, and the boundary condition at position $N+1$ then imposes,

$$\sin[(N+1)\theta] = 0.$$

This last equation in turn gives

$$\theta = \frac{m\pi}{N+1}, \tag{7.5}$$

where m is an integer which labels the modes.

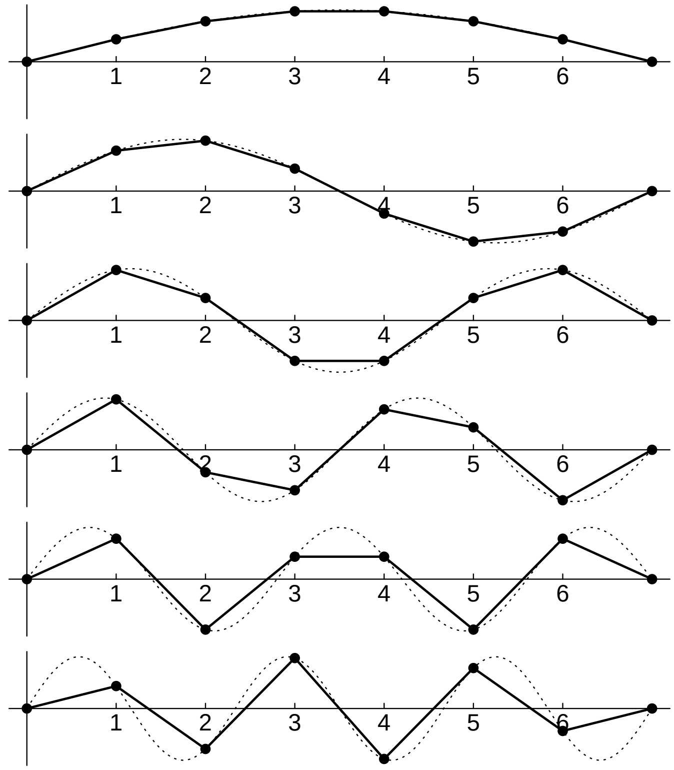

Figure 7.2 The six normal modes of a beaded string fixed at both ends carrying six beads.

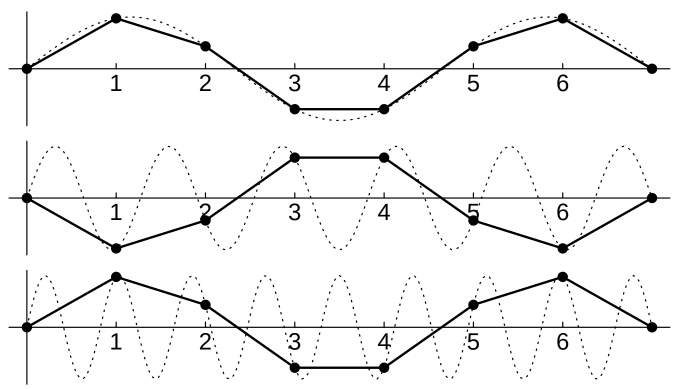

Figure 7.3 Repetition of normal modes for mode numbers greater than six for a string with fixed ends carrying six beads. Modes 3, 11 and 17 are shown. A normal mode remains the same if all the displacements are multiplied by a constant, including -1, so all three modes shown *are* the same.

7.2.3 The Set of Modes

Observe that the linear combination of modes in equation (7.4) is just a sum of left- and right-moving wavelike solutions for the infinite beaded string. For the finite string we are simply constructing a standing wave solution. This is just like finding standing waves for guitar or violin strings or organ pipes, but now the system is discrete rather than continuous.

Look at a string with six beads as an example. There are six degrees of freedom and so we expect six modes as m runs from 1 to 6: these are shown in figure 7.2. The figure also shows the continuous curves obtained by taking n to vary continuously

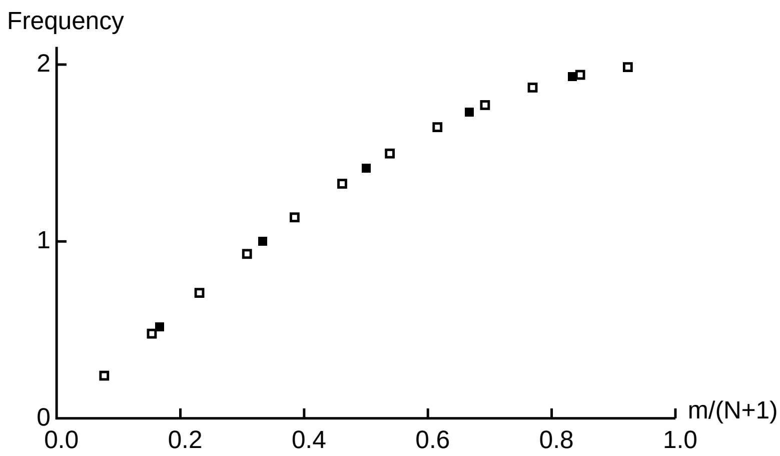

Figure 7.4 Frequencies, in units of $\sqrt{T/Ma}$, of the normal modes of a beaded string with five ($N = 5$, black squares) or twelve ($N = 12$, white squares) beads, showing that the frequencies lie on a universal curve.

and letting na be the position along the string. For larger values of m the modes are repeated (or you get zero displacements). This is shown in figure 7.3. Here you see that the underlying curve of $\sin(n\theta)$ changes, but the positions of the beads, which determine the physical situation are unchanged.

The normal mode frequencies are found by inserting the value of θ from equation (7.5) in equation (7.3) giving ω in terms of θ:

$$\boxed{\omega_m = 2\sqrt{\frac{T}{Ma}}\sin\left(\frac{m\pi}{2(N+1)}\right)}.$$

In figure 7.4 are shown the normal mode frequencies for strings of five ($N = 5$) and twelve ($N = 12$) beads, plotted as functions of $m/(N+1)$. They lie on a universal curve when plotted in terms of this variable. The curve gives the mode frequencies of an infinite line of beads and the finite systems pick out subsets of allowed modes which satisfy the boundary conditions.