A

Supplementary Problems

These are practice questions: you *do not* need to hand in solutions. You can also study past exam papers. PH211 (now PHYS2006) was a new course in 1993, so you'll find some relevant questions in pre 1993 PH101 papers.

1. A rocket burns a kerosene-oxygen mixture: the complete burning of 1 kg of kerosene requires 3.4 kg of oxygen. This burning produces about 4.2×10^7 J of thermal energy. Suppose that *all* of this energy goes into kinetic energy of the reaction products (4.4 kg). What will be the exhaust speed of the reaction products? [Hint: you can answer this by using the expression for the kinetic energy of a system of particles in terms of the centre of mass motion plus that due to the motion relative to the CM.]

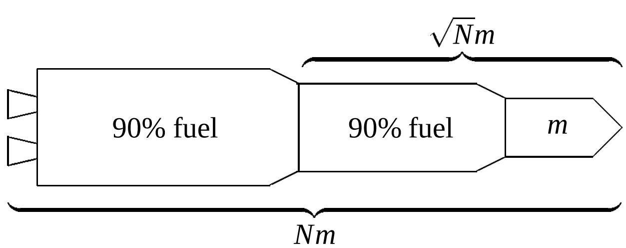
2. At time $t = 0$ a dust particle of mass m_0 starts to fall from rest through a cloud. Its mass grows exponentially with the distance fallen, so that after falling through a distance x its mass is $m_0 \exp(\alpha x)$, where α is a constant. Show that at time t the velocity of the particle is given by

$$v = \sqrt{\frac{g}{\alpha}} \tanh(t\sqrt{\alpha g})$$

where g is the acceleration due to gravity.

3. The total mass of a rocket is 10 kg including fuel. What part of this mass should be fuel in order that the kinetic energy of the rocket after all the fuel is burned is maximised? If the velocity of the exhaust gases is $300\,\mathrm{m\,s^{-1}}$, determine this maximum kinetic energy. Ignore gravity.

4. A payload of mass m is mounted on a two stage rocket. The *total* mass of both rocket stages, fully fuelled, plus the payload, is Nm. The mass of the fully fuelled second stage plus payload is $\sqrt{N}m$. For each stage the exhaust speed is u and the full fuel load makes up 90% of the total mass of the stage.



(i) Show that the speed gained from rest, after first stage burnout and separation followed by second stage burn, is

$$2u\ln\left(\frac{10}{1+9/\sqrt{N}}\right).$$

(ii) If $u = 2.5\,\mathrm{km\,s^{-1}}$, show that the two-stage rocket can achieve a payload velocity of $10\,\mathrm{km\,s^{-1}}$ for large enough N, but that a single stage rocket with the same construction and payload can *never* do so (take the single stage rocket to have payload mass m as before and to have 90% of the stage mass as fuel initially).

5. Find the direction and magnitude of the total torque about the origin produced by the forces $\mathbf{F}_1 = F(\hat{\mathbf{x}} + 2\hat{\mathbf{y}} + \hat{\mathbf{z}})$ acting at $\mathbf{r}_1 = a(\hat{\mathbf{x}} - \hat{\mathbf{y}})$ and $\mathbf{F}_2 = F(2\hat{\mathbf{x}} - 3\hat{\mathbf{y}} + 4\hat{\mathbf{z}})$ acting at $\mathbf{r}_2 = a(\hat{\mathbf{x}} + \hat{\mathbf{z}})$.

6. A car travels round a curve of radius r. If h is the height of the centre of mass above the ground and $2b$ the width between the wheels, show that the car will overturn if the speed exceeds $\sqrt{grb/h}$, assuming no side slipping takes place. If the coefficient of friction between the tyres and the road is μ, show that the car will skid before overturning if $\mu < b/h$.

7. (a) A reel of thread of radius a and moment of inertia Mk^2 is allowed to unwind under gravity, the upper end of the thread being fixed. Find the acceleration of the reel and the tension in the thread.

(b) Find the acceleration of a uniform cylinder of radius a rolling down a slope of inclination θ to the horizontal.

8. (a) A mass M is suspended at a distance ℓ from its centre of mass. By writing down the equation of rotational motion, show that the period of small oscillations is

$$2\pi\sqrt{\frac{I}{mg\ell}}$$

where I is the moment of inertia about the point of suspension.

(b) A body of moment of inertia I about its centre of mass is suspended from that point by a wire which produces a torque τ per unit twist. Show that the period of small oscillations is

$$2\pi\sqrt{I/\tau}$$

9. Calculate the moments of inertia of:

(a) a thin rod about its end

(b) a thin circular disc about its axis

(c) a thin circular disc about its diameter

(d) a thin spherical shell about a diameter

(e) a uniform sphere about a diameter

Note that already known results, together with symmetry, may help you.

10. Two cylinders are mounted upon a common axis and a motor can make one rotate with respect to the other. Otherwise the system is isolated. The following sequence of operations takes place:

(a) one half is rotated with respect to the other through angle ϕ

(b) the moments of inertia of the cylinders change from I_1 and I_2 to I_1' and I_2'

(c) the two halves are rotated back until they are in their original *relative* positions

(d) the moments of inertia are restored to their original values.

Show that the whole system is at rest but has rotated through an angle

$$\phi \frac{I_2'/I_1' - I_2/I_1}{(1+I_2/I_1)(1+I_2'/I_1')}$$

This illustrates how a falling cat can manage to land on its feet.

11. A thin straight rod 20 m long, having a linear density λ of $0.5\,\mathrm{kg\,m^{-1}}$ lies along the y-axis with its centre at the origin. A 2 kg uniform sphere lies on the x-axis with its centre of mass 3 m away from the rod's centre of mass. What gravitational force does the rod exert on the sphere?

12. A planet of mass m moves in an elliptical orbit around a sun of mass M. Its maximum and minimum distances from the sun are $r_{\max}$ and $r_{\min}$.

Show that the total energy of the planet can be written in the form

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

where $\mathbf{L}$ is the angular momentum. Hence show that

$$r_{\max} + r_{\min} = -\frac{GMm}{E}$$

Using conservation of energy, find the maximum and minimum velocity of the planet ($v_{\max}$ and $v_{\min}$).

Assuming Kepler's law relating the period T of the orbit to the semi-major axis of the ellipse, show that

$$T = \frac{\pi(r_{\max} + r_{\min})}{\sqrt{v_{\max}v_{\min}}}$$

13. For motion under a central conservative force, the total energy and the angular momentum $\mathbf{L}$ are conserved. For the special case of an inverse-square law force, such as gravitation or the Coulomb force, with potential energy $V(\mathbf{r}) = -k/r$, we will show that there is a second conserved vector, the Runge-Lenz vector $\mathbf{A}$, given by

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\hat{\mathbf{r}}$$

By considering $\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) = \frac{d}{dt}(r^2)$, or otherwise, show that

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{\mathbf{v}}{r} - \frac{\mathbf{r}\cdot\mathbf{v}}{r^3}\mathbf{r}$$

where $\mathbf{v} = \dot{\mathbf{r}}$. Use the equation of motion to show that

$$\dot{\mathbf{p}} \times \mathbf{L} = -\frac{mk}{r^3}\mathbf{r} \times (\mathbf{r} \times \mathbf{v})$$

Now use the above results to demonstrate that $\dot{\mathbf{A}} = 0$, or $\mathbf{A}$ is conserved.

$\mathbf{A}$ is perpendicular to $\mathbf{L}$ ($\mathbf{A}\cdot\mathbf{L} = 0$), so $\mathbf{A}$ defines a fixed direction in the orbit plane. Let the angle between $\mathbf{r}$ and $\mathbf{A}$ be θ. Take the dot product of $\mathbf{r}$ with $\mathbf{A}$ to show that

$$rA\cos\theta = L^2 - mkr$$

By comparing to the standard equation, $\ell/r = 1 + e\cos\theta$, express the eccentricity e in terms of the length A of the Runge-Lenz vector. Which point of the orbit is $\mathbf{A}$ directed towards?

Hint: the following identities for arbitrary vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, may be useful

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b}) \\ \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \end{aligned}$$

14. **[Hard]** A ballistic rocket is fired from the surface of the Earth with velocity $v < (Rg)^{1/2}$ at an angle α to the vertical. Assuming the equation for its orbit, show that to achieve maximum range, α should be chosen so that $\ell = 2a - R$, where ℓ is the semi-latus rectum, a is the semi-major axis and R is the Earth's radius. Deduce that the maximum range is $2R\theta$ where

$$\sin\theta = \frac{v^2}{2Rg - v^2}$$

15. A locomotive is travelling due North in latitude λ along a straight level track with velocity v. Show that the ratio of the forces on the two rails is approximately

$$1 + \frac{4\omega v h}{ga}\sin\lambda$$

where h is the height of the centre of mass above the rails and $2a$ is the distance between the rails. Calculate this ratio for a speed of $150\,\mathrm{km\,hr^{-1}}$ in latitude 45 deg North, assuming that $h = 2a$. Which rail experiences the larger force?

16. A uniform solid ball has a few turns of light string wound around it. If the end of the string is held steady and the ball allowed to fall under gravity, show that the acceleration of the ball is $5g/7$.

17. A body of moment of inertia I is suspended from a torsion fibre for which the restoring torque per unit angular displacement is T; when the angular velocity of the body is Ω it experiences a retarding torque $k\Omega$. If the top end of the fibre is made to oscillate with angular displacement $\phi_0 \sin\omega t$, where $\omega^2 = T/I$, show that the maximum twist in the fibre is $\phi_0(1 + TI/k^2)^{1/2}$.

18. Two identical masses m are suspended by light strings of length l. The suspension points are distance L apart and a light spring of natural length L and spring constant k connects the two masses. Indicate qualitatively the form of the two normal modes for oscillations in the plane of the strings and spring. Write down the equations of motion for small oscillations of the masses in terms of their horizontal displacements x_1 and x_2 from equilibrium. Find the normal mode frequencies and verify your guess for the ratio x_1/x_2 in the two modes.

19. A large number of identical masses m, arranged in a line at equal intervals a, are joined together by identical springs between neighbours, the springs being such that unit extension requires a force μ. The mass at one end is oscillated along the direction of the line with angular frequency ω. Show that a compressional wave is propagated along the line with wavenumber k given by the expression

$$\omega = \omega_0 \sin(ka/2), \qquad \omega_0 = 2(\mu/m)^{1/2}.$$

What happens if ω is made greater than ω_0?