Classical Mechanics - exercise sheet 2.

1. Initial mass $\quad M_i = 2.75 \times 10^4$ kg.

Fuel burned $\quad M_i - M_f = 34\,\mathrm{kg\,s^{-1}} \times 600\,\mathrm{s} = 20{,}400\,\mathrm{kg}$ (of 24,100 kg available)

$\Rightarrow$ Final mass $\quad M_f = (2.75 - 2.04) \times 10^4\,\mathrm{kg} = 7{,}100\,\mathrm{kg}$.

Exhaust speed $\quad u = 4.58 \times 10^3\,\mathrm{ms^{-1}}$

Initial speed $\quad v_i = 7{,}400\,\mathrm{ms^{-1}}$

$\Rightarrow$ from equation (1.5) derived in notes,

$$v_f = 7400 + 4.58 \times 10^3 \ln \frac{2.75}{0.71}$$

$$\Rightarrow \quad \underline{\underline{v_f = 13{,}600\,\mathrm{ms^{-1}}.}}$$

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2. (a) Rearranging eqn (1.5) derived in the notes,

$$\frac{M_i}{M_f} = \exp \frac{v_f - v_i}{u}$$

$\Rightarrow$ if $v_i = 0$, setting $v_f = u$ gives

$$\frac{M_i}{M_f} = \exp(1) = e.$$

The ratio of propellant to projectile masses is hence $\frac{M_i - M_f}{M_f} = \frac{M_i}{M_f} - 1 = e - 1$
$= 1.7.$

(b) The initial mass and exhaust velocity are defined, so the impulse is maximized by maximizing $M_f v_f$, where as above $v_f = u \ln \frac{M_i}{M_f}$

i.e. $\frac{d}{dM_f} M_f u (\ln M_i - \ln M_f) = 0$

$\Rightarrow \quad u\left(\ln M_i - \ln M_f + M_f\left(\frac{-1}{M_f}\right)\right) = 0$

$\Rightarrow \quad \ln \frac{M_i}{M_f} = 1$ (as in part (a))

$\Rightarrow$ since $M_i = 7\,\text{kg}$, we have M_f = projectile mass $= \frac{1}{e} 7\,\text{kg} = 2.6\,\text{kg}.$

$M_i - M_f$ = propellant mass $= (1 - \frac{1}{e}) 7\,\text{kg} = 4.4\,\text{kg}.$

(For the Hayabusa 2 SCI, the actual values were 2.5 kg and 4.5 kg.)

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3. The sand has the same velocity as the toboggan when it leaves, so there is no impulse, and the situation is the traditional one of a body accelerating down the slope under gravity.

Resolving along the slope, the weight component will be

$$mg \sin 30^\circ = mg/2$$

so the acceleration will be a steady value of $g/2$.

(a) For constant acceleration from rest, the distance travelled will be

$$\frac{1}{2}\frac{g}{2}t^2 = 150\text{ m at the bottom of the slope}$$

$$\Rightarrow \quad t = \sqrt{\frac{4 \times 150\text{ m}}{9.8\text{ ms}^{-2}}} = \underline{\underline{7.8\text{ s.}}}$$

(b) The final speed will be $\frac{g}{2}t = \frac{9.8\text{ ms}^{-2}}{2} \times 7.8\text{ s} = \underline{\underline{38\text{ ms}^{-1}.}}$

(c) If lost at a rate of 3 kg s^{-1}, the 20 kg initial sand load will run out after $20/3 = 6.7$ s, so there will be no sand left when the toboggan reaches the bottom of the slope.

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4. We assume that the gas density remains approximately constant in the neck of the balloon and that the acceleration is uniform over this length. For a pressure P and neck area A,

a) the accelerating force $PA = (\rho A L) a$
where ρ is the gas density, L the neck length, and a the resultant acceleration.
Under uniform acceleration, the exhaust velocity will hence be

$$v_{ex} = \sqrt{2aL} = \sqrt{2\frac{P}{\rho L}L} = \sqrt{2P/\rho}$$

If the density of air at 1 bar is $1.23\,\mathrm{kg\,m^{-3}}$, then at pressure P it will be, for an ideal gas,

$$\rho = 1.23\,\mathrm{kg\,m^{-3}}\; P/10^5\,\mathrm{Pa}$$

$$\Rightarrow \quad v_{ex} = \sqrt{2\frac{P}{1.23\,\mathrm{kg\,m^{-3}}\,P/10^5\,\mathrm{Pa}}} = \sqrt{2\frac{10^5\,\mathrm{Pa}}{1.23\,\mathrm{kg\,m^{-3}}}} = \underline{403\,\mathrm{m\,s^{-1}}}.$$

(Note that this is independent of the pressure of gas in the balloon.)

b) The initial mass of air, at 0.05 bar, will be

$$M_i - M_f = 0.5\,\mathrm{m^3}\;1.23\,\mathrm{kg\,m^{-3}}\;0.05 = 31\,\mathrm{g}$$

Hence, from equation (1.5) of the notes,

$$v_f = 403\,\mathrm{m\,s^{-1}}\,\ln\frac{2.5+31}{2.5} = \underline{\underline{1050\,\mathrm{m\,s^{-1}}}}.$$

c) The ISS has an orbital speed of $7.7\,\mathrm{km\,s^{-1}}$. Assuming this to be typical of the balloon, the velocity of the balloon in the frame of the counter-orbiting satellite will be about twice this, and the kinetic energy hence

$$\tfrac{1}{2}\,0.0025\,\mathrm{kg}\,(2\times 7700\,\mathrm{m\,s^{-1}})^2 = \underline{\underline{300\,\mathrm{kJ}}}.$$

[If the balloon's speed is taken into account, the energy could be 260–340 kJ, depending upon the orientation of the balloon's trajectory with respect to the satellite.]

d) For a cricket ball of speed v to have the same energy,

$$\tfrac{1}{2}\,0.16\,\mathrm{kg}\;v^2 = 300\,\mathrm{kJ} \Rightarrow v = \sqrt{2\frac{300\,000\,\mathrm{J}}{0.16\,\mathrm{kg}}} = \underline{\underline{1900\,\mathrm{m\,s^{-1}}}}.$$

e) If the pressure fell, the exhaust speed would not change, but the exhaust density would fall. The acceleration would thus be reduced but prolonged, and the final speed would be the same. (The assumptions are, of course, rather simplistic.)