Classical Mechanics - exercise sheet 3.

1.(a) Divide the cylinder into cylindrical shells of inner radius r and thickness δr. As above, the moment of inertia will be Mr^2, where $M = \rho 2\pi r \delta r l$, where l is the cylinder length.

The moment of inertia of the solid cylinder will hence be, taking the limit $\delta r \to 0$,

$$\int_0^a \rho 2\pi r l r^2 dr = 2\pi l \rho a^4/4.$$

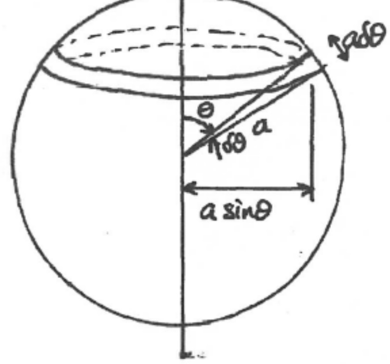
The volume will be $\pi a^2 l$, hence the mass $M = \rho \pi a^2 l$.

The moment of inertia will hence be

$$I = \rho \pi a^2 l \frac{2a^2}{4} = \frac{1}{2} Ma^2.$$

(b) It is helpful to divide the spherical shell into rings, labelled by the angle θ to the axis.

Taking the shell to have thickness δa,



mass of ring $= \rho\, 2\pi a \sin\theta\, a\delta\theta\, \delta a$

$\Rightarrow$ moment of inertia of ring $= \rho\, 2\pi a \sin\theta\, a\delta\theta \delta a (a\sin\theta)^2$

$$\Rightarrow \text{total mass} = \int_0^\pi \rho 2\pi a \sin\theta\, a\, \delta a\, d\theta$$

$$= 2\pi\rho a^2 \delta a \left[-\cos\theta\right]_0^\pi = 4\pi\rho a^2 \delta a = M$$

$$\text{total M of I} = \int_0^\pi \rho 2\pi a \sin\theta\, a \delta a\, a^2 \sin^2\theta\, d\theta$$

$$= 2\pi\rho a^4 \delta a \int_0^\pi \sin\theta (1-\cos^2\theta) d\theta$$

$$= \frac{M}{2} a^2 \left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^\pi = \frac{4}{3}\frac{M}{2}a^2 = \frac{2Ma^2}{3}.$$

(c) We now add spherical shells to form a spherical solid:

$$\text{Mass} = \int_0^a 4\pi\rho r^2 dr = \frac{4\pi\rho a^3}{3} = M$$

$$\text{moment of inertia} = \int_0^a \frac{8\pi\rho r^4}{3} dr = \frac{8\pi\rho a^5}{15}$$

$$= \frac{4\pi\rho a^3}{3} \cdot \frac{2a^2}{5} = \frac{2Ma^2}{5}.$$

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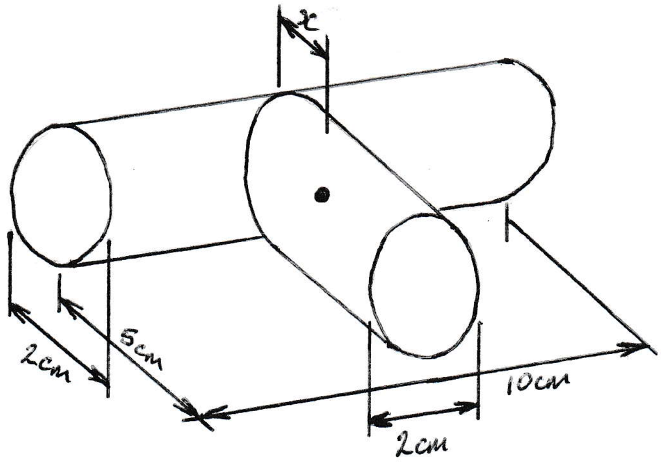
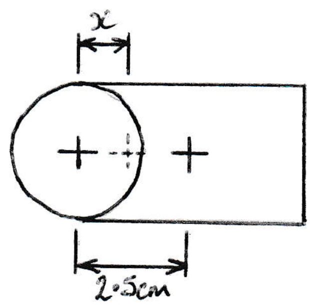


2. Define bar radius $a = 0.01\,m$
bar arm length $l = 0.05\,m$
density $\rho = 8000\,kgm^{-3}$

Neglect material lost where bars overlap.

$\Rightarrow$ mass of long bar $= 2\pi a^2 l \rho$
mass of short bar $= \pi a^2 l \rho$



By symmetry, centre of mass will lie along axis of short bar.
Centre of mass of each bar assumed to lie at centre of each bar.

$\Rightarrow$ displacement x of spanner centre of mass from long bar axis given by

$$x(2\pi a^2 l) = \left(\frac{l}{2} - x\right)(\pi a^2 l)$$

$$\Rightarrow \quad 2x = \frac{l}{2} - x$$

$$\Rightarrow \quad x = \frac{l}{6} = \frac{5}{6}\,cm \approx \underline{0.83\,cm.}$$

If the overlap is taken into account, this will move the centre of mass slightly away from the long bar - ie. $x > 0.83\,cm$.

Classical Mechanics - exercise sheet 3 - cont'd.

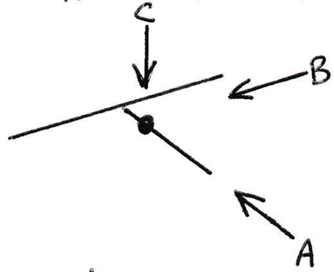
2 cont'd.

We require moment of inertia of solid cylinder about axis $I = \frac{Ma^2}{2}$ from Q1(a)

moment of inertia of thin rod about perpendicular through centre

$$I = \int_{-l/2}^{l/2} \frac{M}{l} x^2 \, dx = \left[\frac{M}{l}\frac{x^3}{3}\right]_{-l/2}^{l/2} \Rightarrow I = \frac{Ml^2}{12}$$

For moments of inertia about axes perpendicular to bar axes, we assume the thin bar result — an underestimate.



Bar masses $M_S = \pi a^2 l \rho$ $M_L = 2\pi a^2 l \rho$

Moment of inertia about axis A:

$$I = \frac{M_S a^2}{2} + \frac{M_L (2l)^2}{12} = \pi a^2 l \rho \left(\frac{a^2}{2} + \frac{8l^2}{12}\right) \quad \underline{\underline{= 2.2 \times 10^{-4} \, kgm^2.}}$$

Moment of inertia about axis B:

$$I = \left(\frac{M_L a^2}{2} + M_L x^2\right) + \int_{-x}^{l-x} \frac{M_S}{l} x'^2 \, dx' \quad \text{(using parallel axis theorem)}$$

$$= \frac{M_L}{2}\left(a^2 + 2\left(\frac{l}{6}\right)^2\right) + \frac{M_S}{3l}\left[x'^3\right]_{-x}^{l-x}$$

$$= \frac{M_L}{2}\left(a^2 + \frac{2l^2}{36}\right) + \frac{M_S}{3l}\left((l-x)^3 + x^3\right)$$

$$= \pi a^2 l \rho \left[a^2 + \frac{2l^2}{36} + \frac{l^2 - 3lx + x^2}{3}\right]$$

$$= \pi a^2 l \rho \left[a^2 + \frac{2l^2}{36} + l^2 \frac{1 - \frac{1}{2} + \frac{1}{36}}{3}\right] \quad = \underline{\underline{8.5 \times 10^{-5} \, kgm^2.}}$$

Classical Mechanics - exercise sheet 3 - cont'd.

2 cont'd.

Moment of Inertia about axis C:

$$I = \left(\frac{M_L (2l)^2}{12} + M_L x^2\right) + \int_{-x}^{l-x} \frac{M_S}{l} x'^2 \, dx' \qquad \text{(using parallel axis theorem)}$$

$$= M_L\left(\frac{l^2}{3} + \left(\frac{l}{6}\right)^2\right) + \frac{M_S}{3l}\left((l-x)^3 + x^3\right) \qquad \text{(as before for final term)}$$

$$= \pi a^2 l \rho \left[2l^2\left(\frac{1}{3} + \frac{1}{36}\right) + l^2 \frac{1 - \frac{1}{2} + \frac{1}{36}}{3}\right] = \underline{\underline{2.8 \times 10^{-4} \text{ kg m}^2.}}$$

(We shall see later in this course that the order $I_B < I_A < I_C$ is significant.)

CLASSICAL MECHANICS - exercise sheet 3.

3. This analysis follows the derivation of the continuous/impulse result for the rocket of section 1.1.3 of the notes, which is valid since m/M is small enough that the change in the truck's momentum with each bullet/shell fired is a small fraction of the total.
Hence, using equation (1.5) of the notes, with $v_i = 0$,

$$v_f = u \ln \frac{M_i}{M_f} = u \ln \frac{M+Nm}{M} = \underline{\underline{u \ln\left(1+\frac{Nm}{M}\right)}}.$$

Had the mass Nm been ejected in one go, conservation of momentum would have required

$$Nmu = Mv_f$$

$$\Rightarrow \quad \underline{\underline{v_f = u\left(\frac{Nm}{M}\right)}} > u\ln\left(1+\frac{Nm}{M}\right) \equiv u\left(\frac{Nm}{M} - \frac{1}{2}\left(\frac{Nm}{M}\right)^2 + \frac{1}{3}\left(\frac{Nm}{M}\right)^3 - \dots\right)$$

(Formal proof – not required: $e^x = 1 + x + x^2/2! + x^3/3! + \dots > 1+x$ for true x

$$\Rightarrow \quad x > \ln(1+x)$$

$$\Rightarrow \quad Nm/M > \ln(1+Nm/M) \;)$$

$\Rightarrow$ it would indeed be better to eject the same mass all at once if possible with the same relative speed.

Classical Mechanics - exercise sheet 3 - cont'd.

4. Gravity acts on the raindrop, so momentum is not conserved, but Newton's second law holds. Assuming the cloud to be stationary,

$$mg = F = \frac{d}{dt}(mv) \underline{\underline{= m\dot{v} + \dot{m}v.}} \qquad (1)$$

For density ρ and drop radius r, $\quad m = \frac{4}{3}\pi r^3 \rho$

and the cross-sectional area will be $\quad A = \pi r^2$

so if the cloud density is ρ_c, the accretion rate will be

$$\dot{m} = (\pi r^2 v)\rho_c$$

$$= \pi v \rho_c \left(\frac{3m}{4\pi\rho}\right)^{2/3}$$

$$= \left(\frac{3}{4\pi\rho}\right)^{2/3} \pi\rho_c\, v\, m^{2/3}$$

which we may write as

$$\underline{\underline{\dot{m} = c\, v\, m^{2/3}}} \qquad (2)$$

where $c \equiv \left(\frac{3}{4\pi\rho}\right)^{2/3} \pi\rho_c$.

As prompted, $\quad \frac{dm}{dx} = \frac{dm/dt}{dx/dt} = \frac{\dot{m}}{v} = c m^{2/3}$

$$\Rightarrow \quad \frac{1}{m^{2/3}}\, dm = c\, dx$$

so, integrating from zero mass at $x=0$,

$$3 m^{1/3} = c x$$

$$\Rightarrow \quad \underline{\underline{m = \left(\frac{cx}{3}\right)^3.}} \qquad (3)$$

Differentiating (3) with respect to time,

$$\dot{m} = \left(\frac{c}{3}\right)^3 3x^2 \dot{x}$$

$$= 3\frac{m}{x} v$$

$$\Rightarrow \quad \frac{\dot{m}}{m} = 3\frac{v}{x}$$

4. cont'd.

We may now use a standard result, which may be derived thus:

$$\dot{v} \equiv \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} = \frac{1}{2}\frac{d}{dx}(v^2)$$

[To see the general significance, apply this to the motion of a mass m:

$$m\dot{v} = \frac{1}{2}\frac{d}{dx}(mv^2)$$

$$\Rightarrow \quad \dot{p} = \frac{d}{dx}\left(\frac{1}{2}mv^2\right) = \frac{d}{dx}(K)$$ where p = momentum, K = kinetic energy

Applying Newton's second law, and assuming conservation of energy $K+V$ = const,

$$F = \dot{p} = -\frac{dV}{dx}$$]

hence, using equation (i),

$$g = v\frac{dv}{dx} + 3\frac{v}{x}v$$

$$= \frac{1}{2}\frac{d}{dx}(v^2) + \frac{3}{x}(v^2)$$ i.e. a differential equation for v^2 in terms of x

rearranging, $$\underline{\underline{\frac{d(v^2)}{dx} + 6\frac{v^2}{x} - 2g = 0.}}$$

Multiplying by x^6, $$2gx^6 = x^6\frac{d(v^2)}{dx} + 6x^5v^2$$

$$= \frac{d}{dx}\left(v^2x^6\right)$$

integrating, $$\frac{2gx^7}{7} = v^2x^6$$

$$\Rightarrow \quad v^2 = \frac{2}{7}gx$$

but $\dot{v} = \frac{1}{2}\frac{d}{dx}(v^2)$

$$\Rightarrow \quad \dot{v} = \frac{1}{2}\frac{d}{dx}\left(\frac{2}{7}gx\right) = \underline{\underline{g/7.}}$$