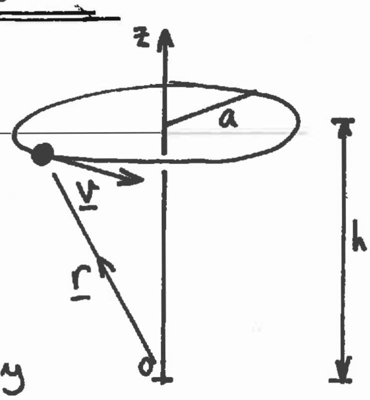
Classical Mechanics – exercise sheet 4.



1\. a) For single particle

$\underline{L} = \underline{r}\times\underline{p} = m\underline{r}\times\underline{v}$ ... which will point

$\Rightarrow$ $\underline{L}$ and $\underline{\omega}$ are not parallel.

[Although not asked to do so, we may calculate $\underline{L}$, taking

$\underline{r} = [a\cos\omega t, a\sin\omega t, h]$

$\Rightarrow \underline{L} = [a\cos\omega t, a\sin\omega t, h] \times m[-a\omega\sin\omega t, a\omega\cos\omega t, 0]$

$= m[-ha^2\omega\cos\omega t, -ha^2\omega\sin\omega t, a^2\omega\cos^2\omega t + a^2\omega\sin^2\omega t]$

$= m\omega a^2[-h\cos\omega t, -h\sin\omega t, 1]$]

b) For a pair of particles, the (x,y) components of $\underline{L}$ will cancel, and the z components add, so the resultant will have only z-components and hence be parallel to $\underline{\omega}$.

[The magnitude of the resultant will be, from above, $2m\omega a^2$.]

2\. a) That moments of inertia can be summed means they are "additive"; they are linear functions of mass (and are calculated by summing the contributions from individual elements of the body, so the additive property is assumed at this stage).

b) Volume of cylinder of radius r and length l is $\pi r^2 l$

$\Rightarrow$ if cylinder of radius a has mass m, cylinder of radius αa will have mass $\alpha^2 m$

$\Rightarrow$ mass of hollow cylinder will be $\underline{\underline{M = m(1-\alpha^2)}}$.

c) The cylinder of radius a has mass m and moment of inertia $\frac{1}{2}ma^2$

$\Rightarrow$ cylinder of radius αa will have a moment of inertia $\frac{1}{2}(\alpha^2 m)(\alpha a)^2 = \frac{1}{2}\alpha^4 ma^2$

$\Rightarrow$ moment of inertia of hollow tube will be $\frac{1}{2}ma^2 - \frac{1}{2}\alpha^4 ma^2 = (1-\alpha^4)ma^2/2$

but $M = m(1-\alpha^2)$

$\Rightarrow$ tube's moment of inertia $= \dfrac{(1-\alpha^4)ma^2}{(1-\alpha^2)m}\dfrac{M}{2} = \underline{\underline{(1+\alpha^2)Ma^2/2.}}$

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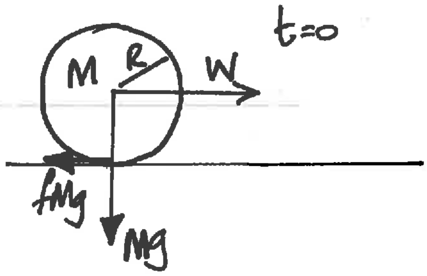
3. Net force on sphere (given that reaction balances weight) will be fMg



$\Rightarrow M\dot{v} = -fMg$ for forward speed v

$\Rightarrow v = W - fgt.$

Torque about centre $= fMgR$

Moment of inertia $= \frac{2}{5}MR^2$

$\Rightarrow \dot{\omega}\,\frac{2}{5}MR^2 = fMgR$ where ω is the angular velocity

$\Rightarrow \dot{\omega} = \frac{5}{2}\frac{fg}{R}$

$\Rightarrow \omega = \frac{5}{2}\frac{fg}{R}t.$

Ball will stop skidding when $v = R\omega$

ie. $W - fgt = \frac{5}{2}fgt$

$\Rightarrow W = \frac{7}{2}fgt$ $\Rightarrow \underline{t = \frac{2}{7}\frac{W}{fg}.}$

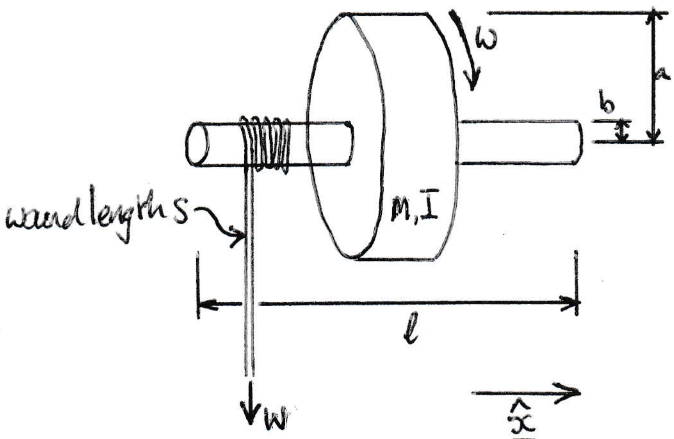
If $W = 2.5\,ms^{-1}$, $f = 0.25$, then $t = \frac{2}{7}\frac{2.5\,ms^{-1}}{0.25 \times 9.8\,ms^{-2}} = \underline{0.29\,m.}$

Exercise Sheet 4 – contd.

4a) Torque from string

$$\tau = Wb = I\dot{\omega}$$

and, for the thin string around axle radius b,



$$s = \int_0^T b\omega \, dt$$

where T is the duration of the starting pull.

Since τ is constant during this period, integration gives

$$\omega = \int_0^t \dot{\omega} \, dt = \int_0^t \frac{\tau}{I} \, dt = \frac{Wb}{I} t$$

$$\Rightarrow \quad s = \int_0^T b \frac{Wb}{I} t \, dt = \frac{1}{2} \frac{Wb^2}{I} T^2$$

$$\Rightarrow \quad T = \sqrt{\frac{2Is}{Wb^2}}$$

and hence $\omega = \frac{Wb}{I} \sqrt{\frac{2Is}{Wb^2}} = \underline{\underline{\sqrt{\frac{2Ws}{I}}}}$.

This may alternatively be derived by noting that the work done Ws is converted into kinetic energy $\frac{1}{2} I\omega^2$.

The angular velocity vector will be parallel to the x-axis, though the direction is not defined in the question – ie.

$$\underline{\underline{\underline{\omega} = \pm \sqrt{\frac{2Ws}{I}} \, \hat{\underline{x}}}}$$

$$\Rightarrow \quad \underline{\underline{\underline{L} = \pm \sqrt{2IWs} \, \hat{\underline{x}}}}.$$

Exercise Sheet 4 - cont'd.

4b) Once released, the weight of the gyroscope will exert a torque

$$\tau_g = mg\frac{l}{2}$$

Since $\underline{\tau}_g = \dot{\underline{L}} = \underline{\Omega} \times \underline{L}$, and $\underline{\tau}$, $\underline{L}$ and $\underline{\Omega}$ will be mutually orthogonal,

$$\Omega L = \tau_g = mg\frac{l}{2}$$

$$\Rightarrow \quad \Omega = \frac{mgl}{2L} = \frac{mgl}{2}\sqrt{\frac{1}{2IWS}} = \underline{\underline{\frac{mgl}{\sqrt{8IWS}}}}$$

c) With the values given,

$$\omega = \sqrt{\frac{2 \times 10N \times 1m}{\frac{1}{2}\,0.1kg(0.03m)^2}} = \underline{\underline{670 \text{ rad } s^{-1}}} \quad (\sim 100 \text{ revs } s^{-1})$$

$$\Omega = \frac{0.1kg\; 9.8ms^{-2}\; 0.03\,m}{\sqrt{\frac{8}{2}\,0.1kg(0.03m)^2\; 10N\; 1m}} = \underline{\underline{0.5 \text{ rad } s^{-1}}}$$