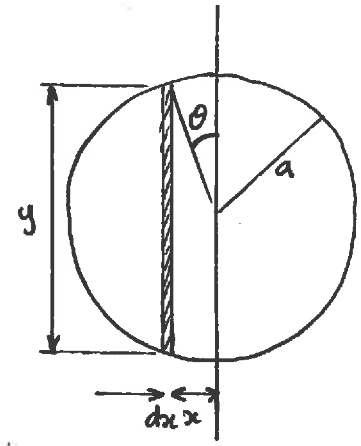
# CLASSICAL MECHANICS – exercise sheet 5.

1. Part (a) is perhaps most easily deduced from the answer to part (b) using the perpendicular axis theorem, but the direct solution is as follows.




(a) Consider the slice shown:

$$\text{area} = y\,dx$$

$\Rightarrow$ if density (mass per unit area) is $\rho$,

mass of element $dm = \rho y\,dx$

moment of inertia $dI = \rho y\,dx\,x^2$

and $(y/2)^2 = a^2 - x^2$

$$\Rightarrow M = \int dm = \int_{-a}^{a} \rho y\,dx = \int_{-a}^{a} \rho\sqrt{4(a^2-x^2)}\,dx = \pi a^2 \rho \text{ (more directly)}$$

$$I = \int dI = \int_{-a}^{a} \rho y x^2\,dx = \int_{-a}^{a} \rho\sqrt{4(a^2-x^2)}\,x^2\,dx$$

It is more convenient to write this as a function of $\theta$, where $x = a\sin\theta$, $y = 2a\cos\theta$, $dx = a\cos\theta\,d\theta$

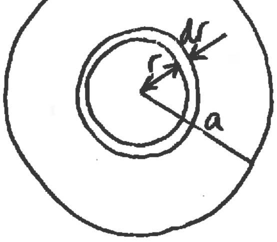
$$\Rightarrow I = \int_{-\pi/2}^{\pi/2} 2\rho\sqrt{a^2 - a^2\sin^2\theta}\;a^2\sin^2\theta\,dx = \int_{-\pi/2}^{\pi/2} 2\rho a^4 \sin^2\theta\cos^2\theta\,d\theta$$

$$= \frac{\rho a^4}{2}\int_{-\pi/2}^{\pi/2} 4\sin^2\theta\cos^2\theta\,d\theta = \frac{\rho a^4}{4}\int_{-\pi}^{\pi}\sin^2 2\theta\,d(2\theta) = \frac{\rho a^4}{4}\frac{2\pi}{2}$$ since $\sin^2 x$ averages $\frac{1}{2}$ over a cycle

$$\Rightarrow \underline{\underline{I = \frac{\pi\rho a^4}{4} = \frac{1}{4}Ma^2.}}$$

(b) Since by symmetry $I_x = I_y$, the perpendicular axis theorem gives directly

$$\underline{\underline{I_z = \frac{1}{4}Ma^2 + \frac{1}{4}Ma^2 = \frac{1}{2}Ma^2.}}$$




This may be derived directly by considering concentric rings

$dm = 2\pi r\,dr\,\rho$ ... hence mass as above

$dI = 2\pi r\,dr\,\rho\,r^2$

$$\Rightarrow I = \int_0^a 2\pi\rho r^3\,dr = 2\pi\rho\left[\frac{r^4}{4}\right]_0^a = \frac{\pi\rho a^4}{2} \underline{\underline{= \frac{1}{2}Ma^2.}}$$

Given this result and the symmetry that $I_x = I_y$, it follows that $I_x = I_y = \frac{1}{2}\left(\frac{1}{2}Ma^2\right) = \underline{\underline{\frac{1}{4}Ma^2.}}$

# CLASSICAL MECHANICS – exercise sheet 5 – cont'd.

2. a) $\underline{L} = \underline{r} \times \underline{p} = \underline{r} \times m\dot{\underline{r}}$

$$= m \begin{vmatrix} i & j & k \\ x_0 + at^2 & bt & ct^3 \\ 2at & b & 3ct^2 \end{vmatrix}$$

$$= m \begin{pmatrix} 3bct^3 - bct^3 \\ 2act^4 - 3ct^2(x_0 + at^2) \\ b(x_0 + at^2) - 2abt^2 \end{pmatrix} = m \begin{pmatrix} 2bct^3 \\ -c(at^4 + 3x_0 t^2) \\ b(x_0 - at^2) \end{pmatrix}$$

$$\underline{\underline{\equiv m\left\{2bct^3 \hat{\underline{i}} - c(at^4 + 3x_0 t^2)\hat{\underline{j}} + b(x_0 - at^2)\hat{\underline{k}}\right\}.}}$$

b) $\underline{F} = m\ddot{\underline{r}} = \underline{\underline{m\left\{2a\hat{\underline{i}} + 6ct\hat{\underline{k}}\right\}.}}$

$$\frac{d\underline{L}}{dt} = m\left\{6bct^2\hat{\underline{i}} - c(4at^3 + 6x_0 t)\hat{\underline{j}} + b(-2at)\hat{\underline{k}}\right\}$$

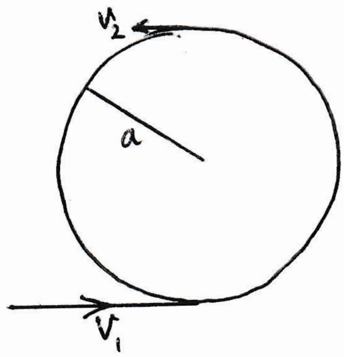
$$\underline{r} \times \underline{F} = m \begin{vmatrix} i & j & k \\ x_0 + at^2 & bt & ct^3 \\ 2a & 0 & 6ct \end{vmatrix} = m \begin{pmatrix} 6bct^2 \\ 2act^3 - 6ct(x_0 + at^2) \\ -2abt \end{pmatrix}$$

$$= m \begin{pmatrix} 6bct^2 \\ -c(4at^3 + 6x_0 t) \\ -2abt \end{pmatrix}$$

$$\equiv m\left\{6bct^2\hat{\underline{i}} - c(4at^3 + 6x_0 t)\hat{\underline{j}} - 2abt\hat{\underline{k}}\right\}$$

$$\Rightarrow \underline{\frac{d\underline{L}}{dt} = \underline{r} \times \underline{F}.}$$

# Classical Mechanics – exercise sheet 5 – contd.




3\. Energy is conserved (since none is lost)

$\Rightarrow \frac{1}{2}Mv_1^2 = \frac{1}{2}Mv_2^2 + Mg2a$

where M is the glider's mass.

Rearranging, and cancelling M, gives

$$v_1^2 = v_2^2 + 4ga$$

To be "weightless" at the top of the loop, the centripetal acceleration $v_2^2/a$ should be provided by gravity, ie.

$$v_2^2/a = g.$$

Combining our expressions,

$$v_1^2 = ag + 4ga = 5ag$$

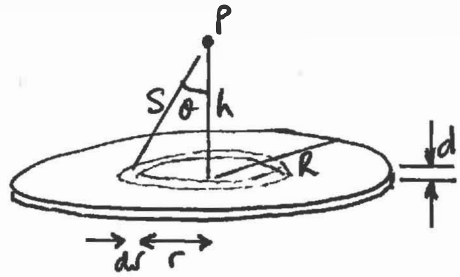
Hence the acceleration at the beginning of the loop will be

$$\frac{v_1^2}{a} = \underline{\underline{5g.}}$$

The wings must produce a lift force to provide this, as well as supporting the glider's weight, so the lift required will be $5mg + mg$ = $\underline{\underline{6 \times \text{glider's weight}}}$ $(mg)$.

# Classical Mechanics – exercise sheet 5 – contd.

4. a) Consider a ring of radius $r$, cross-section $d \times dr$




$\Rightarrow$ mass $dm = 2\pi r\, d\, dr\, \rho$

distance from measurement point P is

$$s = \sqrt{h^2 + r^2}$$

The gravitational forces from each element of the ring will have the same vertical component but horizontal components that cancel, so the total will be vertically downward with magnitude

$$df = \frac{Gm\, dm}{s^2} \cos\theta = \frac{Gm\, 2\pi d\rho\, r\, dr}{h^2 + r^2} \frac{h}{s}$$

$$= Gm 2\pi d\rho h \frac{r\, dr}{(h^2+r^2)^{3/2}}$$

$\Rightarrow$ total force for the disc will be

$$F = \int_{r=0}^{r=R} df = Gm 2\pi d\rho h \int_0^R \frac{r}{(h^2+r^2)^{3/2}} dr$$

$$= Gm 2\pi d\rho h \left[ -(h^2+r^2)^{-1/2} \right]_0^R$$

$$= Gm 2\pi d\rho h \left( \frac{1}{h} - \frac{1}{\sqrt{h^2+R^2}} \right)$$

$$= 2\pi Gm\rho d \left( 1 - \frac{h}{\sqrt{h^2+R^2}} \right)$$

hence the gravitational field $F/m$ will be

$$\underline{g_D = 2\pi G\rho d \left( 1 - \frac{h}{\sqrt{h^2+R^2}} \right)}.$$

# Classical Mechanics - exercise sheet 5 - cont'd.

4.b) If the room is sufficiently large, we may assume the result for $R \gg h$, which gives $\quad g_D = 2\pi G\rho d$

With the values given $\quad g_D = 2\pi \; 6.67\times10^{-11}\,Nm^2kg^{-2}\; 1025\,kg\,m^{-3}\; 0.15\,m$
$$= 6.4\times10^{-8}\;ms^{-2}.$$

The pendulum period is given by $T = 2\pi\sqrt{\frac{l}{g}}$ for pendulum length $l$

$$\Rightarrow \quad \frac{dT}{dg} = 2\pi\, l^{1/2}\left(-\frac{1}{2}g^{-3/2}\right)$$
$$= \frac{-1}{2g}T$$
$$\Rightarrow \quad \frac{dT}{T} = -\frac{1}{2}\frac{dg}{g} = -\frac{1}{2}\;\frac{6.4\times10^{-8}\;ms^{-2}}{9.8\;ms^{-2}} \quad \text{with the values given}$$

In one year, the clock will therefore gain (since the period is reduced)
$$\Delta T = \frac{1}{2}\,\frac{6.4\times10^{-8}}{9.8}\,(365\times24\times3600\;s)$$
$$\underline{\underline{= 0.1\,s.}}$$

c) If the seam of iron ore is wide, we may again assume $R \gg h$ and draw, as above,
$$\Delta T = \frac{1}{2}\,\frac{(3000-2700)\,2\pi\,6.67\times10^{-11}\;120}{9.8}\,(365\times24\times3600)$$
$$\underline{\underline{= 24\;s/year.}}$$

Since the iron ore is denser than the typical Earth's crust, the clock will again run faster, with a shorter oscillation period, and <u>gain</u> 24 s/year.