Classical Mechanics - Exercise Sheet 9.

1. $$\underline{\ddot{r}} = \underline{g}^* - 2\underline{\omega}\times\underline{\dot{r}}$$

integrate wrt time:

$$\underline{\dot{r}} = \underline{v}_0 + \underline{g}^* t - 2\underline{\omega}\times(\underline{r}-\underline{r}_0) \quad \text{where } \underline{r}_0, \underline{v}_0 = \underline{\dot{r}}(0) \text{ are initial values}$$

assume that to first order in ω, we may make the approximation

$$\underline{r} = \underline{r}_0 + \underline{v}_0 t + \tfrac{1}{2}\underline{g}^* t^2 \qquad \text{(motion in an inertial frame)}$$

so that

$$\underline{\dot{r}} = \underline{v}_0 + \underline{g}^* t - 2\underline{\omega}\times(\underline{v}_0 t + \tfrac{1}{2}\underline{g}^* t^2)$$

Integrating again now yields

$$\underline{r} = \underline{v}_0 t + \tfrac{1}{2}\underline{g}^* t^2 - 2(\underline{\omega}\times\underline{v}_0)\frac{t^2}{2} - (\underline{\omega}\times\underline{g}^*)\frac{t^3}{3} + \underline{r}_0$$

so, if $\underline{r}_0 = 0$,

$$\underline{r} = \underline{v}_0 t + \tfrac{1}{2}\underline{g}^* t^2 - (\underline{\omega}\times\underline{v}_0)t^2 - \tfrac{1}{3}(\underline{\omega}\times\underline{g}^*)t^3$$

In assuming $\underline{g}^*$ to be constant, we have again limited our analysis to first order in ω.

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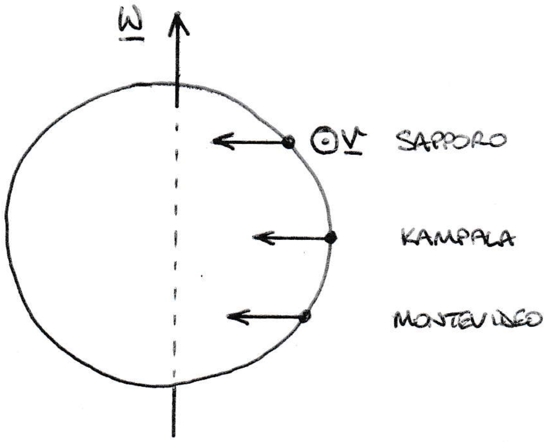
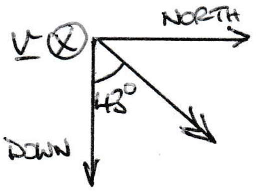
2. In addition to the downward acceleration due to gravity, the bullet will experience a Coriolis acceleration $-2\underline{\omega}\times\underline{v}$, where $\underline{v}$ is of magnitude v and points westward in each location. The Coriolis acceleration is hence in each case of magnitude $2\omega v$, directed perpendicularly in towards the Earth's axis.



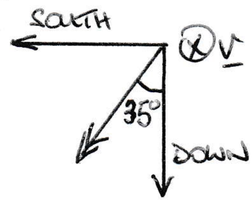
The locally-observed deflections are hence

a) down and to the north / right



b) straight down

c) down and to the south / left



3. From our studies of the Coriolis force, we know that a circular path of constant speed results in the rotating coordinate frame when the path in the inertial frame is a straight track of constant velocity. Here, the ant runs with constant speed in the rotating frame so as to follow a straight line in the inertial frame of the oven. Its path in the turntable frame is hence a circle through the turntable centre.

In the turntable frame, we may equate the Coriolis acceleration to the centripetal acceleration needed to describe the circular path, giving

$$2\omega v = \frac{v^2}{r}$$

where ω is the turntable's angular velocity, v the speed of the ant, r the radius of the circular path, and we have taken the ant's velocity to be in the plane of the turntable, hence perpendicular to its rotation axis.

If the turntable radius is a, the ant will escape it

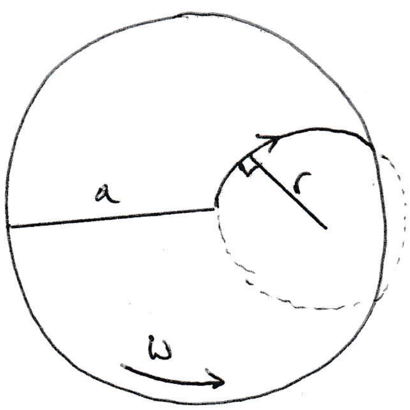
$$2r > a$$

$$\Rightarrow \quad v > a\omega .$$

With the values given, $v = 0.1\,\mathrm{ms^{-1}}$

$$a\omega = 0.15\,\mathrm{m} \times \frac{2\pi}{10\,\mathrm{s}} = 0.09\,\mathrm{ms^{-1}}$$

so v is a little greater than $a\omega$, and the ant escapes.



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4. $g^* = g - \underline{\omega} \times (\underline{\omega} \times \underline{r})$ where we set $\underline{r} = R\hat{\underline{k}}$, $g = -g\hat{\underline{k}}$, $\underline{\omega} = \omega \begin{pmatrix} 0 \\ \cos\lambda \\ \sin\lambda \end{pmatrix}$

so $$g^* = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} - \omega^2 R \begin{pmatrix} 0 \\ \cos\lambda \\ \sin\lambda \end{pmatrix} \times \left[\begin{pmatrix} 0 \\ \cos\lambda \\ \sin\lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} - \omega^2 R \begin{vmatrix} i & j & k \\ 0 & \cos\lambda & \sin\lambda \\ \cos\lambda & 0 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} - \omega^2 R \begin{pmatrix} 0 \\ \sin\lambda\cos\lambda \\ -\cos\lambda\cos\lambda \end{pmatrix}$$

$$= -\begin{pmatrix} 0 \\ \omega^2 R \sin\lambda\cos\lambda \\ g - \omega^2 R\cos^2\lambda \end{pmatrix}$$

$$\Rightarrow |g^*| = \sqrt{(g - \omega^2 R\cos^2\lambda)^2 + (\omega^2 R \sin\lambda\cos\lambda)^2}.$$

If $|g^*|$ doesn't vary with latitude, $|g^*|^2$ must not vary with λ, so $\frac{d|g^*|^2}{d\lambda} = 0$.

$$\Rightarrow \frac{d|g^*|^2}{d\lambda} = \frac{d}{d\lambda}\left\{(g - \omega^2 R\cos^2\lambda)^2 + (\omega^2 R\sin\lambda\cos\lambda)^2\right\} = \frac{d}{d\lambda}\left\{(g - \omega^2 R\cos^2\lambda)^2 + \left(\omega^2 R \cdot \frac{\sin 2\lambda}{2}\right)^2\right\}$$

$$= 2(g - \omega^2 R\cos^2\lambda)(+\omega^2 R\, 2\cos\lambda\sin\lambda) + (\omega^2 R)^2\, 2\, \tfrac{1}{4} \sin 2\lambda\, 2\cos 2\lambda$$

$$= (2g - 2\omega^2 R\cos^2\lambda + \omega^2 R\cos 2\lambda)(\omega^2 R \sin 2\lambda)$$

$$= \left[2g + \omega^2 R(-2\cos^2\lambda + 2\cos^2\lambda - 1)\right](\omega^2 R\sin 2\lambda)$$

$$= \omega^2 R \sin 2\lambda\,(2g - \omega^2 R)$$

$$\Rightarrow g = \frac{\omega^2 R}{2}.$$

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4 contd. Since $g = \dfrac{GM}{R^2}$ where $M = \frac{4}{3}\pi R^3 \rho$,

$$\frac{4}{3}\pi\rho G R = \frac{\omega^2 R}{2}$$

$$\Rightarrow \quad \rho = \frac{3\omega^2}{8\pi G} = \frac{3(2\pi/T)^2}{8\pi G}$$ where T is the rotational period

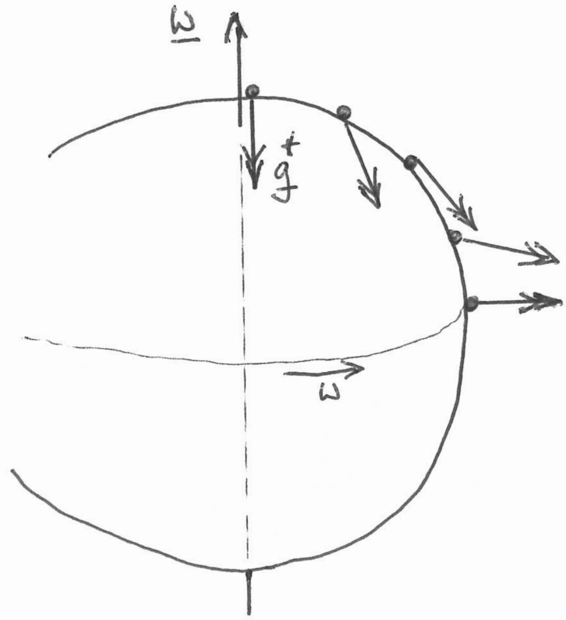
$$= \frac{3\pi}{2GT^2}$$

$$= \frac{3\pi}{2\ 6.67\times10^{-11}\,(2\times60^2+46\times60+40)^2}$$

$$\underline{= 707\ \mathrm{kg\,m^{-3}}.}$$

Although on Isobarios the magnitude of the apparent gravity is constant, its direction relative to the centre of the planet varies markedly. Substituting $\omega^2 R/2 = g$ into our earlier expression for g^*, we obtain

$$g^* = \begin{pmatrix} 0 \\ -2g\sin\lambda\cos\lambda \\ -g+2g\cos^2\lambda \end{pmatrix} \equiv \begin{pmatrix} 0 \\ -g\sin 2\lambda \\ g\cos 2\lambda \end{pmatrix}$$



As the explorers move towards the equator, the surface will appear to slope downwards with increasing steepness until, at 45° latitude, it seems to be a "vertical" cliff face and, by the equator, they need to hang off it as if from a ceiling.