(A1) (Bookwork)

$$T = \frac{1}{2} M\dot{\bar{R}}^2 + \sum_1^N{}_i \frac{1}{2} m_i \dot{\bar{\rho}}_i^2 \qquad \bar{\rho}_i + \bar{R} = \bar{r}_i \quad [1]$$

(position relative to CM)

[1] $M = \sum_1^N{}_i m_i$ total mass

$\dot{\bar{R}}$ = velocity of the CM $\bar{R} = \frac{\sum_1^N{}_i m_i \bar{r}_i}{\sum_1^N{}_i m_i}$ [1] $\bar{r}_i$ position [1]

First term is KE of CM and is frame dependent [1]

Second term is internal KE and is frame independent [1]

(A2) (Bookwork)

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad [2]$$

(A3) (Bookwork)

Force is central when directed along the line joining the object exerting the force and the one subject to it. [1]

Two examples are the gravity force [1] and the electric force. [1]

(A4) 1. The orbits of planets are ellipses with the Sun at one focus [1]

(Bookwork)

2. The radius vector from the Sun to a planet sweeps out equal areas in equal times [1]

3. The square of the orbital period of a planet is proportional to the cube of the semi-major axis of the planet's orbit [1]

(A5) (Bookwork)

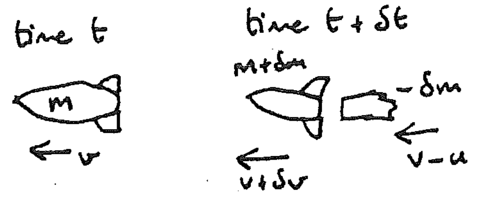
$\bar{\omega} \times \dot{\bar{r}} \Rightarrow$ Coriolis term: apparent force $\perp$ to velocity of particle [2]

$\bar{\omega} \times (\bar{\omega} \times \bar{r}) \Rightarrow$ Centrifugal term: apparent force acting radially outward from rotation axis because of use of rotating frame [2]

$-g \frac{\bar{r}}{r} \Rightarrow$ Gravitational force term: directed towards centre of Earth of magnitude g close to the surface [2]

B1 (a) Momentum of isolated system is conserved.

New Exercise)



[2]

$$(m+\delta m)(v+\delta v) - \delta m\,(v-u) = mv$$

to first order:

$$m\,\delta v + u\,\delta m = 0 \quad [2]$$

$$\frac{dm}{dv} = -\frac{m}{u} \quad [1]$$

Integrate from v_i, m_i to $v_f, m_f \Rightarrow$ $\displaystyle u\int_{m_i}^{m_f}\frac{dm}{m} = -\int_{v_i}^{v_f} dv$

$$\boxed{\Delta v = v_f - v_i = u\ln\frac{m_i}{m_f}} \quad [2] \quad \text{(Book work)}$$

(b) (i) first burn: $m_i = Nm$

$$\left.\begin{array}{l} m_i = Nm \\ m_f = nm + r(Nm - nm) = [rN + n(1-r)]m \end{array}\right\} [2]$$

$$\therefore \boxed{v_1 = u\ln\left(\frac{N}{rN+n(1-r)}\right)} \quad [2]$$

(ii) 2nd burn: $m_i = nm$ [or just let $(N,n)\to(n,1)$ in last result]

$$m_f = m + r(nm - m) = [rn + (1-r)]m \quad [1]$$

$$\therefore \boxed{v_2 = u\ln\left(\frac{n}{rn+1-r}\right)} \quad [2]$$

(iii) $$v_1 + v_2 = u\ln\left[\frac{Nn}{(rN+n(1-r))(rn+1-r)}\right]$$

$$\frac{d(v_1+v_2)}{dn} = u\left(\frac{1}{n} - \frac{1-r}{rN+n(1-r)} - \frac{r}{rn+1-r}\right)$$

[can check that $\left.\dfrac{d^2(v_1+v_2)}{dn^2}\right|_{n=\sqrt{N}} < 0$ — not required]

vanishes when: $$\frac{1}{n} = \frac{1}{n+\frac{r}{1-r}N} + \frac{1}{n+\frac{1-r}{r}} \quad [2]$$

$$\cancel{n^2} + \cancel{n\frac{r}{1-r}N} + \cancel{n\frac{1-r}{r}} + N = \cancel{n^2} + \cancel{n\frac{1-r}{r}} + n^2 + \cancel{n\frac{r}{1-r}N}$$

$$\boxed{n = \sqrt{N}} \quad [2]$$

For this n:

$$v_1 = u\ln\left(\frac{N}{rN+\sqrt{N}(1-r)}\right) = u\ln\left(\frac{\sqrt{N}}{r\sqrt{N}+(1-r)}\right)$$

$$v_2 = u\ln\left(\frac{\sqrt{N}}{r\sqrt{N}+(1-r)}\right) \quad \boxed{\text{equal}} \quad [2]$$

(B2) □ We'll need mom of inertia of disc about axis through its centre, $\perp$ to plane of disc.

(New Exercise)

Let ρ = surface mass density $= \frac{m}{\pi a^2}$ [1]

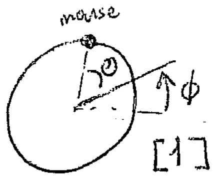
$$I = \int dm\, r^2 = \rho \int_0^a 2\pi r . r^2 dr = \frac{2}{4}\rho \pi a^4$$

dm element of mass [1]   [1] def.n

$$= \frac{1}{2} . 6ma^2 = \boxed{3ma^2}$$ [2]

□ Now let ϕ be anticlockwise angular displacement of disc and θ be angular displacement of mouse rel. to disc.

Then by cons. of ang mom : [2]



[1] for diagram

$$\underset{\text{mouse}}{m(\dot\theta + \dot\phi) r^2} = \underset{\text{disc}}{-3ma^2 \dot\phi}$$ [2]

$$\dot\phi = \frac{-mr^2}{3ma^2 + mr^2}\dot\theta$$ [2]

(i) for path at $r = a$, have $\Delta\theta = \pi$

then $\Delta\phi = -\frac{1}{4} . \pi$ [2]

(ii) for path at $r = \frac{a}{2}$, have $\Delta\theta = -\pi$

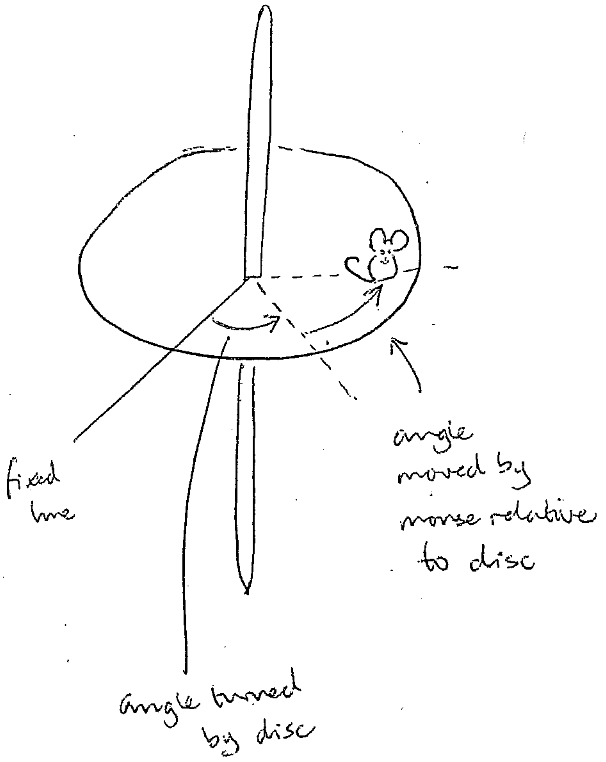
then $\Delta\phi' = \frac{-\frac{1}{4}}{3 + \frac{1}{4}} . (-\pi) = \frac{\pi}{13}$ [2]

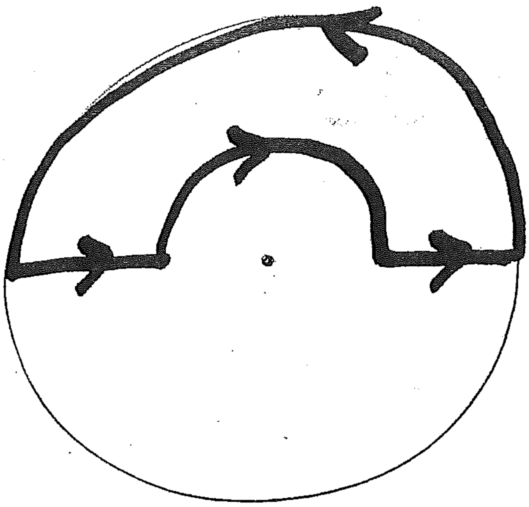
$\Rightarrow$ overall displacement:

$$\Delta\phi + \Delta\phi' = -\pi\left(\frac{1}{4} - \frac{1}{13}\right) = \boxed{\frac{-9\pi}{52}}$$ [2]

$-$sign says disc displacement is <u>clockwise</u>

[2]





MOUSE'S PATH ON DISC

B3 (New exercise)

a)

[1] $F_g = mg$

[1] $F_g = G\dfrac{mM_e}{R_e^2}$ $\Rightarrow$ $g = \dfrac{GM_e}{R_e^2}$ ① [1]

b) In case of circular motion

$r_c = \dfrac{m l_c^2}{k}$ [1] $\qquad k = GM_e m$, $\quad l_c = \dfrac{L}{m}$ ② $m \rightarrow$ mass of spacecraft

Angular momentum of space craft for unit mass

$l_c = r_c v_c$ [1] ③

with v_c speed of spacecraft. Thus

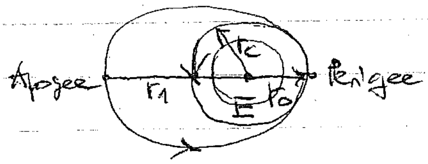
$r_c = \dfrac{m(v_c r_c)^2}{GM_e m}$ [1]

$\Rightarrow v_c^2 = \dfrac{GM_e}{r_c}$ [1] ④

From ① $v_c = \left(\dfrac{gR_e^2}{r_c}\right)^{1/2}$ also. ⑤

c) Low-lying orbit $r_c = R_e$. This radius of the circular orbit is also the perigee distance of the elliptical orbit

$r_0 = r_c = R_e$ [1]



[2] for the graph

Let v_0 be the velocity at perigee required to send the spacecraft to apogee at $r_1 = 60 R_e$.
Because eccentricity of initial circular orbit is zero

$$r_0 = \frac{m l_c^2}{k} \frac{1}{\varepsilon + 1} = \frac{m l_c^2}{k} \qquad (6) \quad [1]$$

But

$$l_c = r_c v_c = r_0 v_c \qquad (r_c = r_0) \qquad (7) \quad [1]$$

Thus, substituting (7) into (6), we have

$$r_0 = \frac{k}{m v_c^2} \qquad (8) \quad [1]$$

After the speed boost from v_c to v_0 at perigee, we obtain an elliptical orbit of eccentricity

$$\varepsilon = \frac{m l_0^2}{k r_0} - 1 = \frac{m v_0^2 r_0}{k} - 1 \qquad (9) \quad [1]$$

since

$$l_0 = r_0 v_0 \qquad (10) \quad [1]$$

Inserting (8) into (9) gives

$$\left(\frac{v_0}{v_c}\right)^2 = \varepsilon + 1 \qquad (11) \quad [1]$$

Now find ε from geometry

$$r_1 = (1 + \varepsilon) \frac{r_1 + r_0}{2} \qquad (12) \quad [1]$$

$$\Rightarrow \varepsilon + 1 = \frac{2 r_1}{r_1 + r_0} \qquad (13) \quad [1]$$

Hence

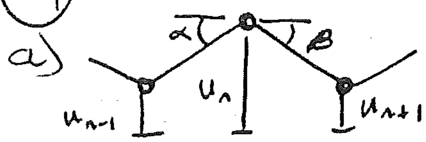
$$\frac{v_0}{v_c} = \sqrt{\frac{2 r_1}{r_1 + r_0}} = \sqrt{\frac{120 R_e}{61 R_e}} \simeq 1.40 \qquad (14) \quad [1]$$

Thus a boost of 40% in speed is required. [1]

(Bookwork)

B4

a)



[1]

$$-M\ddot{u}_n = T(\sin\alpha + \sin\beta) \quad [1]$$

$$\ddot{u}_n = -\frac{T}{Ma}\left(u_n - u_{n-1} + u_n - u_{n+1}\right) \quad \text{For small oscillations.}$$

$$\boxed{\ddot{u}_n = \frac{T}{Ma}\left(u_{n+1} - 2u_n + u_{n-1}\right)} \quad [2]$$

b) To incorporate the right boundary conditions must have $u_0 = u_6 = 0$ [2]

c) Now try a normal mode solution: $u_n = e^{in\theta} e^{i\omega t}$ (based on translation inv'ce for infinite system; apply boundary conditions later)

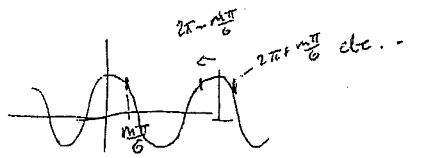
$$-\omega^2 = \frac{T}{Ma}\left(e^{i\theta} - 2 + e^{-i\theta}\right) \Rightarrow \omega^2 = \frac{2T}{Ma}(1 - \cos\theta) \quad [2]$$

Now: $e^{\pm i\theta}$ give same ω, so try: $u_n = \left(A e^{ni\theta} + B e^{-ni\theta}\right) e^{i\omega t}$ [2]

$u_0 = 0 \Rightarrow A = -B$

$u_6 = 0 \Rightarrow A(\sin 6\theta) = 0 \Rightarrow \boxed{\theta = \frac{m\pi}{6}}$ [2]

so: $\boxed{\omega_m^2 = \frac{4T}{Ma}\sin^2\left(\frac{m\pi}{12}\right)}$ [2]



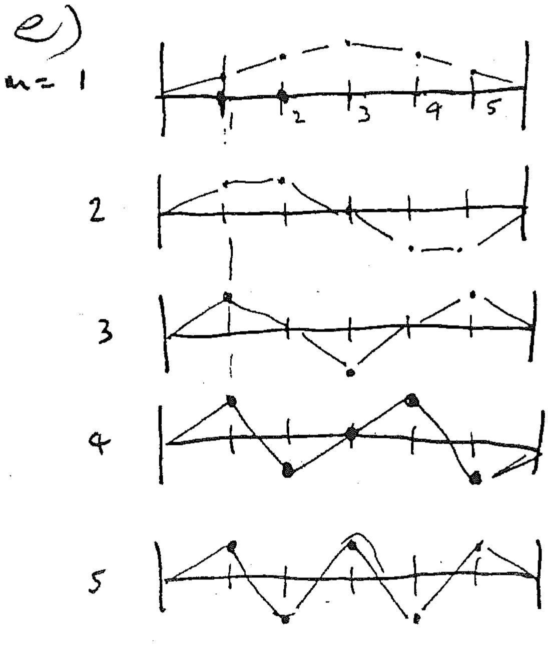
d) now note: if $\cos\left(\frac{m\pi}{6}\right) = \cos\left(\frac{m'\pi}{6}\right)$ then $m' = 12j \pm m$

and $\sin\left(\frac{m'\pi n}{6}\right) = \sin\left(\pm\frac{m\pi n}{6}\right)$ (for displacement of nth bead)

$= \pm\sin\left(\frac{m\pi n}{6}\right)$ → so gives same mode.

∴ only need to count first 5 values of m. [2](*)

e)



[2] for labelling

(*) Also acceptable if student says that if system has n coupled oscillators, hence n degrees of freedom, it always has n normal modes and any additional mode is a superposition of the previous ones.

[2] for shapes