PHYS2006
AY 2012/13

B = bookwork of — i.e. lect notes: discussed in lects, Prob Sheet Qs etc.
(B) = partly bookwork

2001/2002 Sem 1

A1 $\underline{R} = \frac{1}{M}\sum_i m_i \underline{r}_i$ where $M = \sum_i m_i$ [1]B

The 1st term contributing to T can be written $\frac{1}{2}M\dot{R}^2$
This is the kinetic energy of the motion of the centre of mass, [1]
where M is the total mass of the object. [1]
The second term is the kinetic energy of the constituents in the centre of mass frame. It is thus the internal kinetic energy of the system and is the same in all frames. [1]B

(for inertial frames)

A2 The formula for moment of inertia in this case is simply $I = mr^2$ [1]B Since the larger loop has twice
[1]B the mass, and twice the radius [1]B, its moment of inertia is $2^3 = 8$ [1]B times as large.

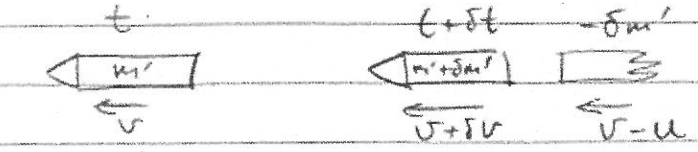
A3 (1) The orbits of the planets are ellipses with the Sun at one focus [1]B
(2) The radius vector from the Sun to a planet sweeps out equal areas in equal times [1]B
(3) The square of the orbital period of a planet is proportional to the cube of the semimajor axis of the planet's orbit [2]B
Only (2) is valid for all central forces. [1]B

A4 The ball lands 15 cm to the East of the tower, [1]B
because the Earth is rotating in this direction [1]B
1st (•) Conservation of angular momentum [1]B shows that the ball has to rotate faster about the Earth's axis [1]B as it moves towards the centre and thus overtakes the tower.
2nd (•) In the rotating frame of the Earth we find a fictitious (or inertial) force called the Coriolis force $-2m\,\underline{\omega}\times\underline{v}$ where $\underline{\omega}$ is angular velocity of the Earth, m the mass of the ball and $\underline{v}$ its velocity. $(-\underline{\omega}\times\underline{v})$ thus points East leading to the deflection [2]B

A5 A normal mode for a system is one in which each part of the system oscillates with the same frequency [1]B, but, in general, different amplitudes [1]B and phases [1]B

A

B1 (a)



[1]B

Conservation of mom^tm of isolated system [1]B $\Rightarrow$

$m'v = (m'+\delta m')(v+\delta v) - \delta m'(v-u)$ [1]B

$\Rightarrow \quad 0 = m'\delta v + \delta m' u$ [1]B (to 1st order)

[1]B $\therefore \quad v = -u\int_M^m \frac{dm'}{m'} = u\ln\left(M/m\right)$ [1]B [1]B

$v=0$ $\quad m'=M$

(b) mom^tm $p = mv$ [1]B $= mu\ln(M/m)$ [1]

When mom^tm is maximised:

$\frac{dp}{dm} = 0 = u\left\{\ln(M/m) - 1\right\}$ [1]B [2]B

i.e. $m = M/e$ [1]B, therefore mass $M-m = \underbrace{M(1-1/e)}$ [1] $\quad 0.632$
should be fuel.

(c) kinetic energy $T = \frac{1}{2}mv^2 = \frac{1}{2}mu^2\ln^2(M/m)$ [1]

K.E. maximised when $\frac{dT}{dm} = 0 = \frac{1}{2}u^2\left\{\ln^2(M/m) - 2\ln(M/m)\right\}$ [2]

Sol^ns $\ln(M/m) = 0 \Rightarrow T = 0$ (minimum) & [1]

$\ln(M/m) = 2 \quad \Rightarrow \quad m = M/e^2$ [1]

Therefore the part of the mass that should be fuel is $M-m = M(1-1/e^2)$ $\quad 0.865$

[1]

B2 MoI $I = \sum_i m_i \, r_{\perp i}^{\,2} = \int_V \rho(\underline{r}) \, r_\perp^{\,2} \, d^3r$ [1]B

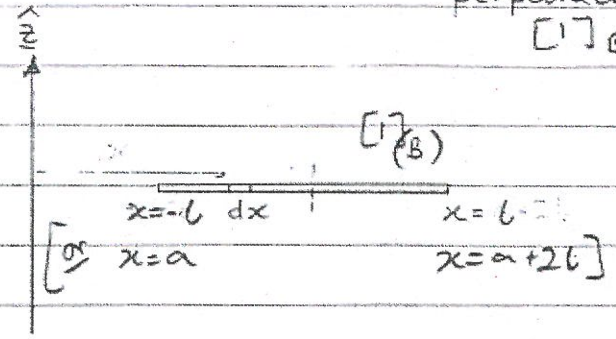
$\rho(\underline{r})$: density [1]B

$r_{\perp i}$, $r_\perp$: perpendicular distance from rot^n axis [1]B [1]B

$\int_V$: integral over body [1]B

m_i: mass of i^{th} particle [1]B

either/or will do.



Let σ = mass/unit length [1]B

$m = 2l\sigma$ [1]B

Either: About centre of rod $I_c = \int_{-l}^{l} dx \, \sigma x^2 = \frac{2}{3}\sigma l^3 = \frac{1}{3} m l^2$ [1]B [1]B [1]B

[1](B) Parallel axis theorem says that if axis displaced a distance $a+l$ from the centre $I = \underbrace{I_c}_{\frac{1}{3}ml^2} + m(a+l)^2$

Or:

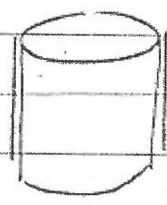
$$I = \int_a^{a+2l} dx \, \sigma x^2 = \frac{\sigma}{3}\left[x^3\right]_a^{a+2l} = \frac{m}{6l}\left\{(a+2l)^3 - a^3\right\}$$ [1] [1] [1]

$$= \frac{m}{3}\left\{3a^2 + 6al + 4l^2\right\} = \frac{m}{3} l^2 + m(a+l)^2$$ [1] [1]

$l = 0.35\text{m}^*$ [1]

$M = 60 - 6 = 54\text{kg}$ [1]

$I = \frac{54}{2}(0.2)^2 = 1.08 \text{ kgm}^2$ [1]



diags + initial data [1]

$I_1 = I + 2\left\{\frac{3}{3}(0.35)^2 + 3(0.2+0.35)^2\right\}$

$= 3.14 \text{ kg m}^2$ [1]

0.1225 (not* 0.49)

0.9075 (not* 2.43)

$I_2 = I + \underbrace{2\times3\times(0.2)^2}_{0.12} = 1.32 \text{ kgm}^2$ [1]

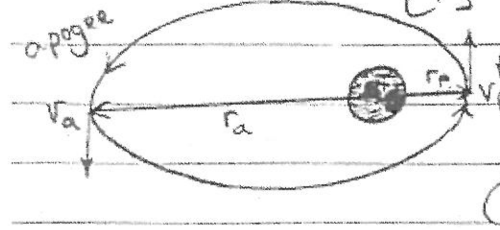
Since angular mom^{tm} is conserved, [1] $I_1\omega_1 = I_2\omega_2$ [1]

leading to a speed up of $\underbrace{\omega_2/\omega_1 = I_1/I_2}_{[1]} = 2.38$ [1]

B3 Relating 2 expressions for gravitational force [1]$_B$

$$\underbrace{mg = \frac{GMm}{r_e^2}}_{[1]_B} \Rightarrow \quad \underline{\underline{g = \frac{GM}{r_e^2}}} \quad [1]_B$$

[1]



Angular mom^m [1](B) consv: $V_p r_p = V_a r_a$ [1](B)

($V_p \perp r_p$ [1](B) & $V_a \perp r_a$, sat' mass cancels)

Consv. Energy [1](B) $\frac{1}{2} V_p^2 - \frac{GM}{r_p} = \frac{1}{2} V_a^2 - \frac{GM}{r_a}$ [1

(sat' mass cancels)

Solve for r_a by eliminating unknown V_a: [1]

$$\frac{1}{2} V_p^2 - \frac{GM}{r_p} = \frac{1}{2} \frac{r_p^2}{r_a^2} V_p^2 - \frac{GM}{r_a} \quad [1]$$

$$\Rightarrow \frac{1}{2} V_p^2 (r_a^2 - r_p^2) = \frac{GM}{r_p}(r_a^2 - r_a r_p)$$

$$\Rightarrow \frac{1}{2} V_p^2 (r_a + r_p) \underset{[1]}{=} \frac{GM}{r_p} r_a = g r_e^2 r_a / r_p \quad [1]$$

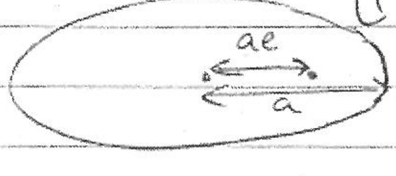
i.e. $$r_a = \frac{\frac{1}{2} V_p^2 r_p}{g r_e^2 / r_p - \frac{1}{2} V_p^2} = \frac{r_p}{\frac{2 g r_e^2}{r_p V_p^2} - 1} \quad [1]$$

Putting in numerical values: $r_p = 6570 \times 10^3$ m

$g = 9.81$ ms^{-2} $\quad r_e = 6370 \times 10^3$ m

$V_p = 8 \times 10^3$ ms^{-1}

[1]



$$\Rightarrow r_a = r_p / 0.8934 = \underline{\underline{7350 \text{ km}}} \quad [1]$$

$r_p = a(1-e)$

$r_a = a(1+e)$ [2]

So $r_a + r_p = 2a \quad r_a - r_p = 2ae$

& $$e = \frac{r_a - r_p}{r_a + r_p} \quad [1]$$

$$= \frac{735 - 657}{735 + 657}$$

$$= \underline{\underline{0.0560}} \quad [1]$$

B4 [1st part is very close to bookwork but not quite; specializing to posⁿ $\underline{x}=0$ at $t=0$ & ignoring $g^* = g + O(\omega^2)$]

Integrate $\ddot{\underline{x}} = \underline{g}^* - 2\underline{\omega}\times\dot{\underline{x}}$ once w.r.t. t ⇒

$$\dot{\underline{x}} = \underline{g}^* t - 2\underline{\omega}\times\underline{x} + \underline{v} \quad [2]$$

(using initial condⁿˢ $\dot{\underline{x}} = \underline{v}$ & $\underline{x} = 0$ at $t=0$). [1]

Now work to 1st order in ω [1] since small [1] $\left(\omega = \frac{2\pi}{24\times 3600}\ \text{rad s}^{-1}\right)$,
Then in $\underline{\omega}\times\underline{x}$ can substitute 0th order solⁿ: [1]

$$\underline{x} = \underline{v}t + \tfrac{1}{2}\underline{g}^* t^2 \quad [1]$$

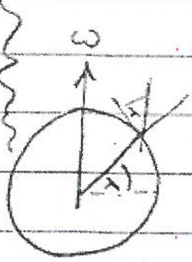
$$\Rightarrow \dot{\underline{x}} = \underline{g}^* t - 2\underline{\omega}\times\underline{v}t - \underline{\omega}\times\underline{g}^* t^2 + \underline{v} \quad [1]$$

Integrate w.r.t. t again & using $\underline{x} = 0$ @ $t=0$: [1]

$$\underline{x} = \underline{v}t + \tfrac{1}{2}\underline{g}^* t^2 - \underline{\omega}\times\underline{v}t^2 - \tfrac{1}{3}\underline{\omega}\times\underline{g}^* t^3$$

Initial condⁿˢ $\underline{x} = 0$ @ $t=0$

Use coords $\hat{x}$ = East, $\hat{y}$ = North, $\hat{z}$ along vertical i.e. along $-\underline{g}^*$ to $O(\omega)$. [1] Diag & [1]



$$\dot{\underline{x}} = \begin{pmatrix} v\cos\alpha \\ 0 \\ v\sin\alpha \end{pmatrix} [1] \quad @\ t=0$$

$$\underline{g}^* = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} [1], \quad \underline{\omega} = \omega\begin{pmatrix} 0 \\ \cos\lambda \\ \sin\lambda \end{pmatrix} [1]$$

$$\underline{\omega}\times\underline{g}^* = \omega g\begin{pmatrix} -\cos\lambda \\ 0 \\ 0 \end{pmatrix} [1] \qquad \underline{\omega}\times\underline{v} = \omega v\begin{pmatrix} \cos\lambda\sin\alpha \\ \sin\lambda\cos\alpha \\ -\cos\lambda\cos\alpha \end{pmatrix} [1]$$

i.e. $$\underline{x} = vt\begin{pmatrix} \cos\alpha \\ 0 \\ \sin\alpha \end{pmatrix} - \tfrac{1}{2}gt^2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \tfrac{1}{3}g\omega t^3\begin{pmatrix} \cos\lambda \\ 0 \\ 0 \end{pmatrix} - \omega v t^2\begin{pmatrix} \cos\lambda\sin\alpha \\ \sin\lambda\cos\alpha \\ -\cos\lambda\cos\alpha \end{pmatrix}$$

Shell strikes ground when $vt\sin\alpha - \frac{1}{2}gt^2 + \omega v t^2\cos\lambda\cos\alpha = 0$ [1]

$$\Rightarrow t = \frac{2v\sin\alpha}{g} + O(\omega) \quad [1]$$

Thus NS deflection is: $y = -\omega v t^2\sin\lambda\cos\alpha$ [1] $= -\dfrac{4\omega v^3}{g^2}\sin\lambda\sin^2\alpha\cos\alpha$

Sign for Northern hemisphere ($\lambda > 0$) is −ve ⇒ South [1]