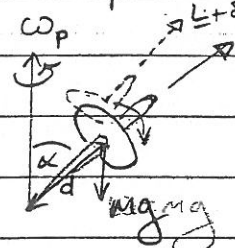
2013/14 B = Bookwork

A1

ω_p does not depend on α [1]$_B$



$$\left.\begin{array}{ll}\text{Torque} & \tau = \text{"} r \times F \text{"} \\ \text{thus} & \tau = dmg\sin\alpha\end{array}\right\} [1]_B$$

[1]$_B$ for diag and/or def^{ns} d

In a small time δt

$$\left.\begin{array}{l}\delta L := |\delta \underline{L}| = L\sin\alpha\ \omega_p \delta t \\ \therefore \dfrac{dL}{dt} = \omega_p L \sin\alpha \\ \tau = dL/dt \Rightarrow \omega_p = \dfrac{dmg}{L}\end{array}\right\} [1]_B$$

A2

Inside the sphere $g(\underline{r}) = 0$ [½]$_B$

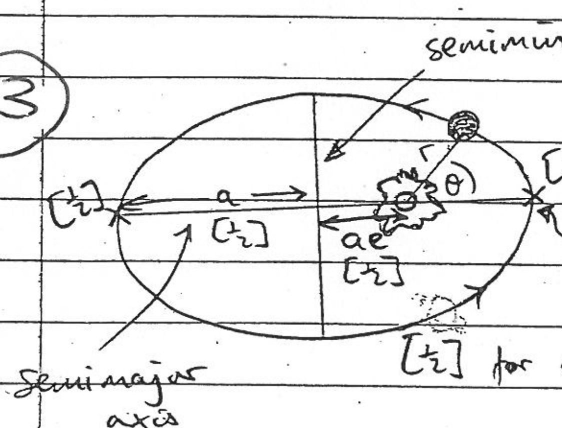
Outside $g(\underline{r}) = -GM\hat{\underline{r}}/r^2$

[½]$_B$ [½]$_B$ [½]$_B$ [½]$_B$ [½]$_B$ [½]$_B$

where G is Newton's gravitational constant and $\hat{\underline{r}}$ is the unit vector $\underline{r}/r$ in the direction $\underline{r}$.

[½]$_B$

A3



x = spots where $\dot{r} = 0$

$\dot{\theta}$ is a maximum here. [1]

[½] for ellipse

B

(A4) $\underline{R}$ is the position of the point in Southampton relative to the centre of the Earth, using axes which rotate with the Earth. [1]B

[1]B ω is the vector angular velocity of the Earth (and thus points north and has magnitude ...)

[1]B $-2\underline{\omega}\times\dot{\underline{r}}\, m$, where m is the mass of the ball, is the Coriolis force. It is proportional to the balls velocity $\dot{r}$ and acts perpendicular to it.

For mostly complete answers

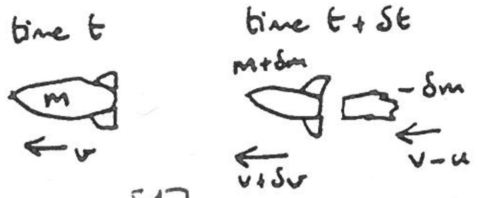
[1]B $m\underline{\omega}\times(\underline{\omega}\times\underline{R})$ is the centrifugal force due to the Earth spinning. In total $g\frac{\underline{R}}{R} + \underline{\omega}\times(\underline{\omega}\times\underline{R}) = g^*$ ~~giving~~ an apparent gravity pointing slightly more towards the equator and with slightly smaller magnitude than g.

(A5) IF $x(t)$ is a time-dependent solution of the harmonic motion, then Time Translation Invariance states that also $x(t+c)$ is a solution, for any constant c

[2]B

[2]B

B1 (a) Momentum of isolated system is conserved. [1]



$(m+\delta m)(v+\delta v) - \delta m\,(v-u) = mv$

to first order: [1]

$m\,\delta v + u\,\delta m = 0$

[1]

[for identifying masses & speeds] $\dfrac{dm}{dv} = -\dfrac{m}{u}$ [1]

Integrate from v_i, m_i to $v_f, m_f \Rightarrow$ $u\displaystyle\int_{m_i}^{m_f}\frac{dm}{m} = -\int_{v_i}^{v_f} dv$ [2]

$$\boxed{\Delta v = v_f - v_i = u \ln \frac{m_i}{m_f}}$$ [1] (Book work)

(b) (i) first burn: $m_i = Nm$

$m_f = nm + r(Nm - nm) = [rN + n(1-r)]m$ [2]

$$\therefore \boxed{v_1 = u \ln\left(\frac{N}{rN + n(1-r)}\right)}$$ [2]

(ii) 2nd burn: $m_i = nm$ [or just let $(N, n) \to (n, 1)$ in last result]

$m_f = m + r(nm - m) = [rn + (1-r)]m$ [1]

$$\therefore \boxed{v_2 = u \ln\left(\frac{n}{rn + 1 - r}\right)}$$ [2]

(iii) $v_1 + v_2 = u \ln\left[\dfrac{Nn}{(rN + n(1-r))(rn + 1 - r)}\right]$ [2]

$$\frac{d(v_1+v_2)}{dn} = u\left(\frac{1}{n} - \frac{1-r}{rN + n(1-r)} - \frac{r}{rn + 1 - r}\right)$$

[can check that $\left.\dfrac{d^2(v_1+v_2)}{dn^2}\right|_{n=\sqrt{N}} < 0$ — not required]

vanishes when: $\dfrac{1}{n} = \dfrac{1}{n + \frac{r}{1-r}N} + \dfrac{1}{n + \frac{1-r}{r}}$

$$\cancel{n^2} + \cancel{n\frac{r}{1-r}N} + \cancel{n\frac{1-r}{r}} + N = \cancel{n^2} + \cancel{n\frac{1-r}{r}} + n^2 + \cancel{n\frac{r}{1-r}N}$$

$$\boxed{n = \sqrt{N}}$$ [2]

For this n:

$$v_1 = u \ln\left(\frac{N}{rN + \sqrt{N}(1-r)}\right) = u \ln\left(\frac{\sqrt{N}}{r\sqrt{N} + (1-r)}\right)$$

$$v_2 = u \ln\left(\frac{\sqrt{N}}{r\sqrt{N} + (1-r)}\right)$$ $\Rightarrow$ $\boxed{\text{equal}}$ [2]

(B2) (a) MoI about a fixed axis

$$I = \sum_i m_i R_i^2$$

R_i = perp dist from rotation axis

mass of ith particle

[2]$_B$

MoI of sphere about a diameter. Let axis be ~~[illegible]~~ x, y, z axis in turn.

$$I_x = \int (y^2+z^2)\,\rho\, d^3r$$

[2]$_B$
$$I_y = \int (z^2+x^2)\,\rho\, d^3r$$
$$I_z = \int (x^2+y^2)\,\rho\, d^3r$$

but $I_x = I_y = I_z$ by symmetry

$$\Rightarrow 3I = 2\rho \int (x^2+y^2+z^2)\, d^3r$$
$$= 2\rho \int_0^a 4\pi r^2 . r^2\, dr \quad [2]_B$$
$$= \frac{8\pi\rho a^5}{5}$$

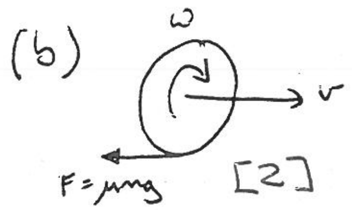
but $m = \frac{4}{3}\pi\rho a^3$

[2]

$$\Rightarrow I = \frac{8\pi}{15}\rho a^5 = \boxed{\frac{2}{5} m a^2}$$

[2]$_B$

(b)



[2]

let ball have radius a, mass m.

linear motion: $m\dot{v} = -\mu m g$ [2]

so: $\boxed{v = u - \mu g t}$ using $v = u$ at $t=0$

angular motion (about CM) [2]

$$I\dot{\omega} = \mu m g a$$

$$\boxed{\omega = \frac{\mu m g a}{I} t}$$ using $\omega = 0$ at $t=0$

[2]

Stops skidding when $v = a\omega$, so

[2]

$$\frac{\mu m g a^2}{I} t = u - \mu g t$$

$$u = \mu g (t)\left(1 + \frac{5ma^2}{2ma^2}\right) \Rightarrow \boxed{t = \frac{2u}{7\mu g}}$$ [2]

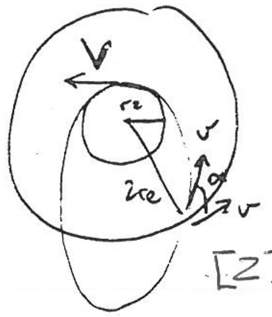
(B3) The gravitational attraction is a central force $\therefore$ it exerts no torque about the Earth. So, the ang. mom of the ~~p~~ satellite wrt the Earth is constant. [2] B

Vector $\underline{L}$ is conserved and is always perp to $\underline{r}$ and $\underline{p}$ $\Rightarrow$ orbit always lies in a plane. [2] B

Other conserved quantities? central forces are conservative so have total energy conserved.

[bonus marks if mention Runge-Lenz Vector]

[2] B



[2]

Initial orbit: $\frac{mv^2}{2r_e} = \frac{GMm}{4r_e^2} \Rightarrow \boxed{v^2 = \frac{GM}{2r_e}}$

[2]

After the instantaneous change.

Consider conserved qties.

□ ang mom: [1] $mVr_e = mv\cos\alpha .\, 2r_e$ (1) [2]

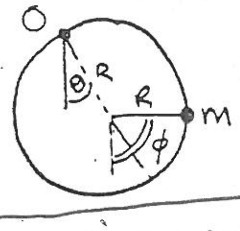
□ energy [1] $\frac{1}{2}mV^2 - \frac{GMm}{r_e} = \frac{1}{2}mv^2 - \frac{GMm}{2r_e}$ (2) [2]

$$\frac{V^2}{2} = \frac{v^2}{2} + \frac{GM}{2r_e} = \frac{3v^2}{2} \Rightarrow V = \sqrt{3}v$$

sub in (1): $V = 2v\cos\alpha$ [2]

$\Rightarrow \cos\alpha = \frac{\sqrt{3}}{2} \Rightarrow \boxed{\alpha = \frac{\pi}{6}}$ [2]

(B4) In a normal mode, every part of the system oscillates with the same frequency. [2]B



let O be pt. of suspension and use coordinates θ, ϕ as shown.

Lagrangian method (1)

Kinetic energy: $T_{ring} = \frac{1}{2} I_0 \dot\theta^2 = \frac{1}{2}(mR^2 + mR^2)\dot\theta^2$

MoI about O ↑ ; ↑ ∥el axis thm

$= \boxed{mR^2\dot\theta^2}$ [2](1)

For bead, horizontal displacement, x, is $R(\theta+\phi)$ for small displacements
vertical displacement is 2nd order in θ, ϕ.

$$\therefore\ T_{bead} = \frac{1}{2} m \dot x^2 = \boxed{\frac{1}{2} mR^2(\dot\theta+\dot\phi)^2}$$ [2](1)

Potential energy:
$$V = -mgR\cos\theta - mgR(\cos\theta + \cos\phi)$$
$$= \boxed{mgR\left(\theta^2 + \frac{\phi^2}{2}\right) + \text{const}} \text{ for small } \theta, \phi.$$ [2](1)

Lagrangian:
$$L = \frac{1}{2}mR^2(3\dot\theta^2 + 2\dot\theta\dot\phi + \dot\phi^2) - mgR\left(\theta^2 + \frac{\phi^2}{2}\right) + \text{const}.$$

E-L eqns: [2]

for θ: $\frac{d}{dt}\left(\frac{1}{2}mR^2[6\dot\theta + 2\dot\phi]\right) = -mgR.2\theta$

for ϕ: $\frac{d}{dt}\left(\frac{1}{2}mR^2[2\dot\theta + 2\dot\phi]\right) = -mgR\,\phi$

$$\Rightarrow \boxed{\begin{pmatrix}3 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\ddot\theta\\ \ddot\phi\end{pmatrix} = -\frac{g}{R}\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\theta\\ \phi\end{pmatrix}}$$

Look for modes $\begin{pmatrix}\theta\\ \phi\end{pmatrix} = \begin{pmatrix}A\\ B\end{pmatrix}e^{i\omega t}$ [2] $\Rightarrow \begin{pmatrix}3-2\alpha & 1\\ 1 & 1-\alpha\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$

solve: $(3-2\alpha)(1-\alpha) - 1 = 0$
$2\alpha^2 - 5\alpha + 2 = 0$ $\Longrightarrow \alpha = \frac{5 \pm 3}{4}$; $\alpha = 2, \frac{1}{2}$ or [2]

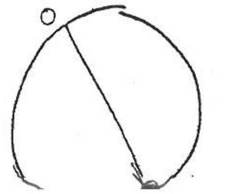
$$\boxed{\omega = \sqrt{\frac{g}{2R}},\ \sqrt{\frac{2g}{R}}}$$

[Note: Students can also apply Newton's laws directly if they wish – I do not require Lagrangian method in this question, though some students know it]

Form of modes

(1) $\omega = \sqrt{\frac{g}{2R}}$ or $\alpha = 2$ [2]

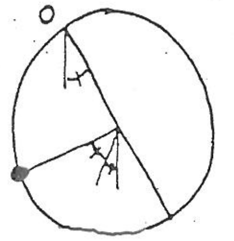
$\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix} = 0$ so $A = B$



[1] bead remains at end of diameter through O (ang freq. of ring alone is same as simple pendulum of length $2R$)

(2) $\omega = \sqrt{\frac{2g}{R}}$ or $\alpha = \frac{1}{2}$ [2]

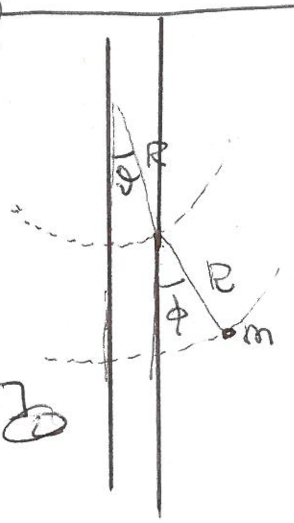
$\begin{pmatrix}2 & 1\\ 1 & 1/2\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix} = 0$ so $B = -2A$



– horizontal displacement of bead and CM of ring are equal and opposite. [1]

Using Newton's laws, method ②

Bead:



[1] ②

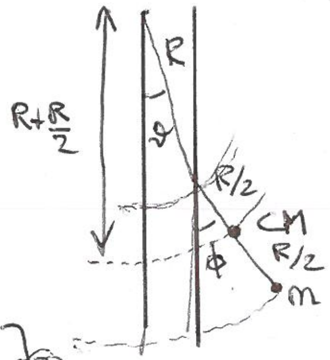
horizontal displacement (small osc.)

$$R\vartheta + R\phi$$ [2] ②

Eq motion

$$mR(\ddot{\vartheta} + \ddot{\phi}) = -mg\phi$$ [1] ②

CM



[1] ②

horizontal displacement (small osc.)

$$\left(R+\frac{R}{2}\right)\vartheta + \frac{R}{2}\phi$$ [2] ②

Eq. motion

$$\cancel{2}m\frac{1}{\cancel{2}}R(3\ddot{\vartheta} + \ddot{\phi}) = -2mg\vartheta$$ [1] ②