SEMESTER 2 EXAMINATION 2015-2016

CLASSICAL MECHANICS

Duration: 120 MINS (2 hours)

This paper contains 9 questions.

Answers to Section A and Section B must be in separate answer books

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language Word to Word® translation dictionary (paper version) is permitted provided it contains no notes, additions or annotations.

Section A

A1. Explain what are meant by the *centre of mass* and *moment of inertia*. Give expressions for both properties, for a system comprising a number of particles with masses m_i and positions $\mathbf{r}_i$. [4]

The centre of mass $\mathbf{r}_{\mathrm{CM}}$ is the point within an object that would follow the same trajectory if replaced by a pointlike particle with the same mass as the object and subject to the same external forces. *[1]*

$$\mathbf{r}_{\mathrm{CM}} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i}.$$ *[1]*

The moment of inertia I is a measure of the object's resistance to changes in rotational motion; it is the constant of proportionality between the angular momentum and angular velocity, and between the torque and angular acceleration. *[1]*

The moment of inertia about a single axis $\hat{\mathbf{n}}$ is given by

$$I_{\mathbf{n}} = \sum_i m_i R_i^2 = \sum_i m_i \left|\hat{\mathbf{n}} \times \mathbf{r}_i\right|^2$$

where R_i is the perpendicular distance of the ith particle from the rotation axis. *[1]*

A2. A uniform solid disc of radius a has a hole of radius b bored axially through its centre to form a ring of mass M. Show that the moment of inertia I of the ring about its axis of rotational symmetry is

$$I = \frac{1}{2} M \left(a^2 + b^2\right).$$ [4]

We may divide the ring into thin-walled tubes of radius r, radial thickness $\mathrm{d}r$ and density ρ per unit area, each of which will have a moment of inertia

$$I(r)\mathrm{d}r = (\rho 2\pi r \mathrm{d}r)\, r^2.$$ *[1]*

The total moment of inertia is found by integrating this from the inside to the outside of the ring:

$$I = \int_b^a \rho 2\pi r^3 \mathrm{d}r = 2\pi\rho \left[\frac{r^4}{4}\right]_b^a = \frac{\pi\rho}{2}\left(a^4 - b^4\right) = \frac{\pi\rho}{2}\left(a^2 - b^2\right)\left(a^2 + b^2\right).$$ *[1]*

The total mass is similarly found to be

$$M = \int_b^a \rho 2\pi r \mathrm{d}r = 2\pi\rho \left[\frac{r^2}{2}\right]_b^a = \pi\rho\left(a^2 - b^2\right).$$ *[1]*

The moment of inertia is hence

$$I = \frac{1}{2} M \left(a^2 + b^2\right).$$ *[1]*

A3. A cricketer strikes a ball with a bat of mass m and moment of inertia I (about its centre of mass), and imparts an impulse Δp. Show that, if no impulse is to be felt at the handle, the change in the angular velocity of the bat must be $\Delta\omega = \Delta p/(mD)$, where D is the distance from the handle to the bat's centre of mass. [2]

If no impulse is felt at the handle, the change in momentum of the bat will equal the impulse imparted by contact with the ball. Hence, if Δv is the change in centre-of-mass speed of the bat,

$$m\Delta v = \Delta p. \qquad [\,1\,]$$

If the speed at the handle is unchanged, the angular velocity of the bat must hence change by

$$\Delta\omega = \Delta v/D = \Delta p/(mD). \qquad [\,1\,]$$

Hence show that the ball should strike the bat a distance $D + I/(mD)$ from the handle – a point known as the *centre of percussion*. [2]

[Assume the angular velocity and moment of inertia to be about the same axis, perpendicular to the plane defined by the handle, point of impact and impulse.]

The bat's angular momentum must hence change by

$$\Delta L = I\Delta\omega = R\Delta p \qquad [\,1\,]$$

where R is the distance of the impulse from the centre of mass. The ball must hence strike the bat a distance

$$D + R = D + I\frac{\Delta\omega}{\Delta p} = D + \frac{I}{mD} \qquad [\,1\,]$$

from the handle – the sign being that which rotates the bat's centre of mass about the handle in the same direction as the impulse.

A4. The magnitude g of the acceleration due to gravity is found to be greater down a mine than it is on the Earth's surface. Show that this can be explained, taking the Earth to have spherical symmetry, if

$$\rho_{\rm s} < \frac{2}{3}\rho_{\rm av}$$

where $\rho_{\rm av}$ is the average density of the Earth and $\rho_{\rm s}$ its density at the surface. [4]

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For a spherically symmetrical mass distribution, the gravitational attraction will be equal to that of the mass M, acting at the centre of the Earth, which lies within a sphere whose surface passes through the measurement point. Hence, if the Earth's radius is R and the distance from its centre to the measurement point is r, the mass will be

$$M = \frac{4}{3}\pi R^3 \rho_{\mathrm{av}} - 4\pi R^2 (R - r)\rho_{\mathrm{s}}. \qquad [\,1\,]$$

The local value of g will be

$$g = \frac{GM}{r^2} \qquad [\,0.5\,]$$

and hence its rate of change with height will be

$$\frac{\mathrm{d}g}{\mathrm{d}r} = -2\frac{GM}{r^3} + \frac{G}{r^2}\frac{\mathrm{d}M}{\mathrm{d}r} = \frac{G}{r^2}\left(\frac{\mathrm{d}M}{\mathrm{d}r} - \frac{2M}{r}\right) \qquad [\,1\,]$$

which, at the Earth's surface, will be

$$\frac{G}{R^2}\left(4\pi R^2 \rho_{\mathrm{s}} - \frac{2}{R}\frac{4}{3}\pi R^3 \rho_{\mathrm{av}}\right) = 4\pi G\left(\rho_{\mathrm{s}} - \frac{2}{3}\rho_{\mathrm{av}}\right), \qquad [\,1\,]$$

which is negative – i.e. g decreases with ascent – if

$$\rho_{\mathrm{s}} < \frac{2}{3}\rho_{\mathrm{av}}. \qquad [\,0.5\,]$$

A5. Show that, if a spacecraft of total mass $m(t)$ propels itself by ejecting exhaust gas from its rocket motor with a relative velocity $\mathbf{u}$, then its velocity $\mathbf{v}(t)$ satisfies

$$m\mathrm{d}\mathbf{v} = -\mathbf{u}\mathrm{d}m,$$ [2]

and hence, making clear any assumptions in your derivation, that the initial and final speeds v_i and v_f are related to the initial and final masses m_i and m_f by

$$v_f = v_i + u \ln \frac{m_i}{m_f}.$$ [2]

Equating the total momenta of the spacecraft and exhaust before and after ejection of an infinitessimal mass $\mathrm{d}m$ that results in a velocity increase $\mathrm{d}\mathbf{v}$,

$$m\mathbf{v} = (m - \mathrm{d}m)(\mathbf{v} + \mathrm{d}\mathbf{v}) + \mathrm{d}m(\mathbf{v} + \mathbf{u}).$$ *[1]*

Expanding this expression, cancelling terms, and neglecting the term $\mathrm{d}m\mathrm{d}\mathbf{v}$, which will be of vanishing significance for infinitessimal changes, we obtain

$$m\mathrm{d}\mathbf{v} = -\mathbf{u}\mathrm{d}m.$$ *[1]*

Assuming $\mathbf{u}$ and $\mathbf{v}$ to be aligned, this expression may be rearranged to give

$$\mathrm{d}v = -u\frac{\mathrm{d}m}{m}$$ *[1]*

which can be integrated to give

$$v_f - v_i = -u\left(\ln m_f - \ln m_i\right)$$

hence

$$v_f = v_i + u \ln \frac{m_i}{m_f}.$$ *[1]*

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Section B

B1. Show that, if a fixed-length vector $\mathbf{A}$ rotates with angular velocity ω about an axis defined by the vector $\hat{\omega}$, and we define $\boldsymbol{\omega} \equiv \omega\hat{\omega}$, then

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \boldsymbol{\omega} \times \mathbf{A}.$$ [4]

From a suitable diagram, we see that the infinitessimal change $\mathrm{d}\mathbf{A}$ resulting from rotation of $\mathbf{A}$ through an infinitessimal angle $\mathrm{d}\varphi$ about $\hat{\omega}$ will be

$$\mathrm{d}\mathbf{A} = \hat{\omega} \times \mathbf{A}\mathrm{d}\varphi.$$ *[2]*

Dividing by an infintessimal timestep $\mathrm{d}t$ and noting that the angular velocity $\omega \equiv \mathrm{d}\varphi/\mathrm{d}t$,

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \hat{\omega} \times \mathbf{A}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \omega\hat{\omega} \times \mathbf{A} = \boldsymbol{\omega} \times \mathbf{A}.$$ *[2]*

The unit vectors $\hat{\mathbf{i}}'$, $\hat{\mathbf{j}}'$ and $\hat{\mathbf{k}}'$ of a rotating coordinate frame rotate with angular velocity ω about an axis $\hat{\omega}$, so that a vector $\mathbf{a} \equiv a_i\hat{\mathbf{i}} + a_j\hat{\mathbf{j}} + a_k\hat{\mathbf{k}}$ in an inertial frame $\left\{\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}\right\}$ may be written at a given time as $\mathbf{b} \equiv b_i\hat{\mathbf{i}}' + b_j\hat{\mathbf{j}}' + b_k\hat{\mathbf{k}}'$. Show that

$$\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} = \dot{\mathbf{b}} + \boldsymbol{\omega} \times \mathbf{b}$$

and hence that

$$\frac{\mathrm{d}^2\mathbf{a}}{\mathrm{d}t^2} = \ddot{\mathbf{b}} + 2\boldsymbol{\omega} \times \dot{\mathbf{b}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}),$$

where $\dot{\mathbf{b}} \equiv \dot{b}_i\hat{\mathbf{i}}' + \dot{b}_j\hat{\mathbf{j}}' + \dot{b}_k\hat{\mathbf{k}}'$, $\ddot{\mathbf{b}} \equiv \ddot{b}_i\hat{\mathbf{i}}' + \ddot{b}_j\hat{\mathbf{j}}' + \ddot{b}_k\hat{\mathbf{k}}'$, and $\dot{b}_i \equiv \mathrm{d}b_i/\mathrm{d}t$ etc. [6]

In an inertial frame, the unit vectors of the rotating frame change with time, so the vector must be differentiated as a product [1 mark per line]: *[3]*

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} &= \left(\frac{\mathrm{d}b_i}{\mathrm{d}t}\hat{\mathbf{i}}' + b_i\frac{\mathrm{d}\hat{\mathbf{i}}'}{\mathrm{d}t}\right) + \left(\frac{\mathrm{d}b_j}{\mathrm{d}t}\hat{\mathbf{j}}' + b_j\frac{\mathrm{d}\hat{\mathbf{j}}'}{\mathrm{d}t}\right) + \left(\frac{\mathrm{d}b_k}{\mathrm{d}t}\hat{\mathbf{k}}' + b_k\frac{\mathrm{d}\hat{\mathbf{k}}'}{\mathrm{d}t}\right) \\
&= \dot{b}_i\hat{\mathbf{i}}' + \dot{b}_j\hat{\mathbf{j}}' + \dot{b}_k\hat{\mathbf{k}}' + b_i\boldsymbol{\omega} \times \hat{\mathbf{i}}' + b_j\boldsymbol{\omega} \times \hat{\mathbf{j}}' + b_k\boldsymbol{\omega} \times \hat{\mathbf{k}}' \\
&= \dot{b}_i\hat{\mathbf{i}}' + \dot{b}_j\hat{\mathbf{j}}' + \dot{b}_k\hat{\mathbf{k}}' + \boldsymbol{\omega} \times \mathbf{b} \equiv \dot{\mathbf{b}} + \boldsymbol{\omega} \times \mathbf{b}.
\end{aligned}$$

Differentiating a second time, noting that the vectors $\mathbf{a}$ and $\mathbf{b}$ are equivalent [1 mark per line], *[3]*

$$\begin{aligned}
\frac{\mathrm{d}^2\mathbf{a}}{\mathrm{d}t^2} &= \left(\ddot{b}_i\hat{\mathbf{i}}' + \dot{b}_i\frac{\mathrm{d}\hat{\mathbf{i}}'}{\mathrm{d}t}\right) + \left(\ddot{b}_j\hat{\mathbf{j}}' + \dot{b}_i\frac{\mathrm{d}\hat{\mathbf{i}}'}{\mathrm{d}t}\right) + \left(\ddot{b}_k\hat{\mathbf{k}}' + \dot{b}_i\frac{\mathrm{d}\hat{\mathbf{i}}'}{\mathrm{d}t}\right) + \boldsymbol{\omega} \times \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} \\
&= \ddot{b}_i\hat{\mathbf{i}}' + \ddot{b}_j\hat{\mathbf{j}}' + \ddot{b}_k\hat{\mathbf{k}}' + \dot{b}_i\boldsymbol{\omega} \times \hat{\mathbf{i}}' + \dot{b}_j\boldsymbol{\omega} \times \hat{\mathbf{j}}' + \dot{b}_k\boldsymbol{\omega} \times \hat{\mathbf{k}}' + \boldsymbol{\omega} \times \left(\dot{\mathbf{b}} + \boldsymbol{\omega} \times \mathbf{b}\right) \\
&= \ddot{\mathbf{b}} + \boldsymbol{\omega} \times \dot{\mathbf{b}} + \boldsymbol{\omega} \times \dot{\mathbf{b}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) = \ddot{\mathbf{b}} + 2\boldsymbol{\omega} \times \dot{\mathbf{b}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}).
\end{aligned}$$

Hence show that, for a particle of mass m subject to gravitational acceleration $\mathbf{g}$ and an applied force $\mathbf{F}$, the equation of motion in the rotating frame will be

$$m\ddot{\mathbf{b}} = \mathbf{F} + m\mathbf{g} - \underbrace{2m\boldsymbol{\omega} \times \dot{\mathbf{b}}}_{*} - \underbrace{m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})}_{**}.$$ [2]

According to Newton's second law, the total force $\mathbf{F} + m\mathbf{g} = m\mathrm{d}^2\mathbf{a}/\mathrm{d}t^2$. *Substituting the result above and rearranging,* *[2]*

$$\begin{aligned} \mathbf{F} + m\mathbf{g} &= m\frac{\mathrm{d}^2\mathbf{a}}{\mathrm{d}t^2} = m\ddot{\mathbf{b}} + 2m\boldsymbol{\omega} \times \dot{\mathbf{b}} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) \\ \Rightarrow \quad m\ddot{\mathbf{b}} &= \mathbf{F} + m\mathbf{g} - 2m\boldsymbol{\omega} \times \dot{\mathbf{b}} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}). \end{aligned}$$

Explain the significance of the terms marked * and **, and why that marked ** may generally be neglected when the axes are referred to the Earth's surface. [3]

The first term is the Coriolis force [0.5] due to conservation of angular momentum as the particle moves relative to the rotating surface [0.5]. The second term is the centrifugal force [0.5] caused by the centripetal acceleration required to prevent the particle from flying off in a straight line [0.5]. The Earth is an oblate spheroid, whose surface is roughly perpendicular to the combination of gravity and centrifugal force, so local axes are aligned to the resultant force, which includes both terms in the local 'gravity' value [1]. *[3]*

Show that if wind is to flow undeflected over the surface of the Earth, then there must be a sideways pressure gradient given by the geostrophic wind condition

$$\frac{\mathrm{d}P}{\mathrm{d}y} = 2\rho\omega \sin(\alpha)\, v$$

where ρ is the air density, v the wind speed, α the latitude and ω the angular velocity at which the Earth rotates. [3]

For undeflected flow, $\ddot{\mathbf{b}} = 0$ *[1] and hence, assuming the centrifugal term to be combined into the local gravity as described above, we find [1]*

$$\mathbf{F} + m\mathbf{g} = 2m\boldsymbol{\omega} \times \dot{\mathbf{b}},$$ *[2]*

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where $\mathbf{F} = -\nabla P$ results from a gradient in atmospheric pressure P. The vertical component accounts for the hydrostatic reduction in pressure with altitude, while the sideways (y) component gives, per unit cross-sectional area, [1]

$$\frac{\mathrm{d}P}{\mathrm{d}y} = 2\rho\,\hat{\mathbf{j}}' \cdot \left(\boldsymbol{\omega} \times \dot{\mathbf{b}}\right) \equiv 2\rho\boldsymbol{\omega} \cdot (\dot{\mathbf{b}} \times \hat{\mathbf{j}}') = 2\rho\,\omega\,v\,\sin\alpha. \qquad [\,1\,]$$

Calculate the horizontal distance over which a pressure drop of 400 Pa is required for a geostrophic wind speed of 13 m s^{-1} at a latitude of 50° N at sea level. The density of air at sea level may be taken to be 1.225 kg m^{-3}. [2]

Rearranging the above expression with a pressure difference ΔP over distance Δy,

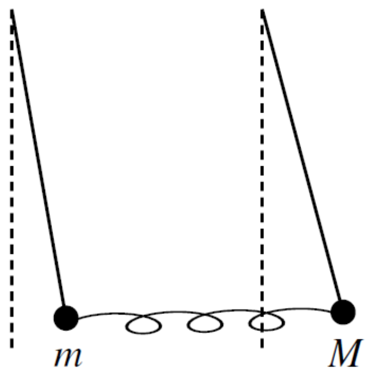
$$\Delta y = \frac{\Delta P}{2\rho\,\omega\,v\sin\alpha} = \frac{400\,\mathrm{Pa}}{2 \times 1.225\,\mathrm{kg\,m^{-3}}\,\frac{2\pi}{3600\,24\,\mathrm{s}}\,13\,\mathrm{m\,s^{-1}}\,\sin\,50^\circ} = 225\,\mathrm{km}. \qquad [\,2\,]$$

[This, happily, is about what is shown on Met. Office synoptic charts.]

B2. (a) Explain what is meant by the *normal mode* of an oscillating system. [2]

A normal mode is a motion in which all parts of the system oscillate with the same single frequency and (therefore) with a fixed phase relationship between each other. *[2]*

(b) Two simple pendulums, each of length l, have bobs of masses m and M. The pendulums are coupled by a weak spring of spring constant k, as shown in the diagram below.



If the displacements of the pendulum bobs are x and X, show that, for small displacements, the motion of the system may be described by

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{pmatrix} x \\ X \end{pmatrix} = \begin{pmatrix} -\frac{g}{l} - \frac{k}{m} & \frac{k}{m} \\ \frac{k}{M} & -\frac{g}{l} - \frac{k}{M} \end{pmatrix}\begin{pmatrix} x \\ X \end{pmatrix}.$$ [6]

We describe the angles of the pendulums by ϑ_m and ϑ_M, the positions of the pendulum bobs by x and X, and assume small ϑ_m and ϑ_M so that $\cos\vartheta_{m,M} \approx 1$. The equations of motion are hence *[2]*

$$\begin{aligned} m\ddot{x} &= -mg\sin\vartheta_m + k(X - x) \\ M\ddot{X} &= -Mg\sin\vartheta_M - k(X - x) \end{aligned}$$

Writing $\sin\vartheta_m = x/l$, $\sin\vartheta_M = X/l$, *[2]*

$$\begin{aligned} m\ddot{x} &= -m(g/l)x + k(X - x) \\ M\ddot{X} &= -M(g/l)X - k(X - x) \end{aligned}$$

so that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{pmatrix} x \\ X \end{pmatrix} = \begin{pmatrix} -\frac{g}{l} - \frac{k}{m} & \frac{k}{m} \\ \frac{k}{M} & -\frac{g}{l} - \frac{k}{M} \end{pmatrix}\begin{pmatrix} x \\ X \end{pmatrix}$$ *[2]*

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(c) Consider the normal modes of this system and find

(i) the corresponding eigenfrequencies; and [5]

(ii) the corresponding eigenvectors. [3]

(i) For a normal mode, we may write

$$\begin{pmatrix} x \\ X \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \exp \mathrm{i}\omega t.$$ *[1]*

Substituting this into the equation of motion above, we obtain

$$\begin{pmatrix} \omega^2 - \frac{g}{l} - \frac{k}{m} & \frac{k}{m} \\ \frac{k}{M} & \omega^2 - \frac{g}{l} - \frac{k}{M} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$ *[1]*

hence *[1]*

$$\begin{aligned} \left(\omega^2 - \frac{g}{l} - \frac{k}{m}\right)\left(\omega^2 - \frac{g}{l} - \frac{k}{M}\right) - \frac{k}{m}\frac{k}{M} &= 0 \\ \Rightarrow \quad \left(\omega^2 - \frac{g}{l}\right)\left(\omega^2 - \frac{g}{l} - \frac{k}{m} - \frac{k}{M}\right) = 0 \end{aligned}$$

from which we obtain the two possible solutions *[2]*

$$\begin{aligned} \omega^2 &= \frac{g}{l} \\ \text{or} \quad \omega^2 &= \frac{g}{l} + \frac{k}{m} + \frac{k}{M}. \end{aligned}$$

(ii) When $\omega^2 = g/l$, we find

$$\begin{pmatrix} -\frac{k}{m} & \frac{k}{m} \\ \frac{k}{M} & -\frac{k}{M} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$ *[0.5]*

for which the solution is $a = b$ and hence the unnormalized eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. *[1]*

When $\omega^2 = g/l + k/m + k/M$, we find

$$\begin{pmatrix} \frac{k}{M} & \frac{k}{m} \\ \frac{k}{M} & \frac{k}{m} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$ *[0.5]*

for which the solution is $a/M = -b/m$ and hence the unnormalized eigenvector is $\begin{pmatrix} M \\ -m \end{pmatrix}$. *[1]*

(d) The system is released from rest with m in its equilibrium position and M displaced a small distance d directly away from the other pendulum. Find an expression describing the subsequent motion. [4]

If we write $\omega_1^2 = g/l, \omega_2^2 = g/l + k/m + k/M$, the general solution for the motion will be of the form

$$\begin{pmatrix} a \\ b \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_1 t + \varphi_1) + \beta \begin{pmatrix} M \\ -m \end{pmatrix} \cos(\omega_2 t + \varphi_2). \qquad [\,1\,]$$

Setting the initial conditions $x = 0$, $X = d$, $\dot{x} = \dot{X} = 0$ for $t = 0$, we obtain

$$\begin{aligned} \varphi_1 &= 0 \\ \varphi_2 &= 0 \\ \alpha + M\beta &= 0 \\ \alpha - m\beta &= d \end{aligned}$$

from which

$$\begin{aligned} \alpha &= \frac{d}{1 + \frac{m}{M}} \\ \beta &= \frac{-d}{M + m} \end{aligned}$$

hence *[2]*

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{d}{1 + \frac{m}{M}} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_1 t) - \frac{1}{M} \begin{pmatrix} M \\ -m \end{pmatrix} \cos(\omega_1 t) \right]$$

or

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{d\,M}{M + m} \begin{pmatrix} \cos(\omega_1 t) - \cos(\omega_2 t) \\ \cos(\omega_1 t) + \frac{m}{M} \cos(\omega_2 t) \end{pmatrix}. \qquad [\,1\,]$$

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B3. State the relationship between torque and angular momentum, and explain what is meant by *precession* in the context of rotational motion. Give an example of precession, and state the physical principle from which it results. [6]

The torque $\boldsymbol{\tau}$ applied to a system is equal to the rate of change of its angular momentum $\mathbf{L}$,

$$\boldsymbol{\tau} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}.$$ *[2]*

Precession is the variation of the axis of rotation of a spinning body [1] due to the application of a torque about a different axis [1]. It is apparent in the toppling of a spinning top, the coupling of atomic angular momenta, and the precession of the Earth's axis (precession of the equinoxes) around the ecliptic pole [1]. It results from the conservation of angular momentum [1] in a system in which the precessing object is coupled to other rotational motions. *[4]*

A disc of mass M spins with angular velocity ω about a light axle along its axis of rotational symmetry, about which the disc has a moment of inertia I. The combination is suspended by attaching the axle, at a distance a from the disc's centre of mass, to a rigid support, and the axle assumes a constant angle α to the vertical. Show that the moment of the disc's weight about the support is

$$Mg\,a\sin\alpha$$

and that the spinning disc precesses about the support with angular frequency

$$\Omega = \frac{Mg\,a}{I\omega}.$$ [4]

If $\mathbf{a}$ is the vector from the pivot to the centre of mass, then the moment of the weight Mg about the pivot will be [the vector form is not required for the mark]

$$|\mathbf{a} \times M\mathbf{g}| = a\sin\alpha Mg.$$ *[1]*

If the spinning disc precesses with angular frequency Ω (ω_p in the figure below), then

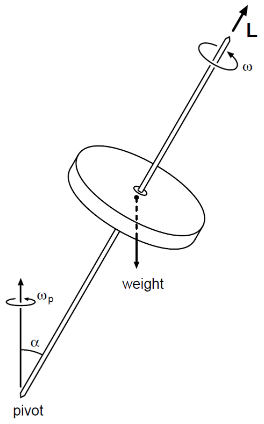
$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \boldsymbol{\Omega} \times \mathbf{L} = \Omega L\sin\alpha$$ *[1]*

where $L = I\omega$ is the angular momentum of the spinning disc. Since the moment of the weight is the torque applied to the spinning disc, this equals $\mathrm{d}\mathbf{L}/\mathrm{d}t$, so

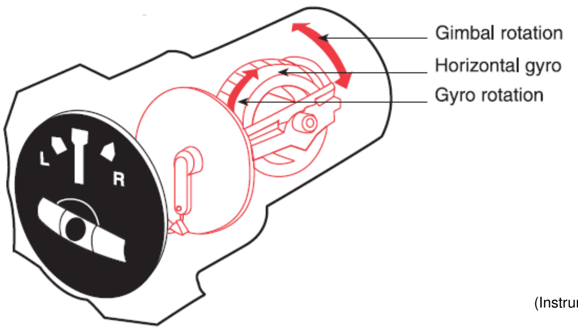
$$Mg\,a\sin\alpha = \Omega L\sin\alpha = \Omega I\omega\sin\alpha$$ *[1]*

from which

$$\Omega = \frac{Mg\,a}{I\omega}.$$ *[1]*



An aircraft's rate-of-turn indicator comprises a gyroscope whose rotation axis during straight flight lies athwart the aircraft (i.e. horizontally from left to right), as shown below. The gyroscope spins within a spring-loaded gimbal that allows it to rotate about the longitudinal (fore-aft) axis of the aircraft, and a pointer linked to the gimbal indicates this rotation. If the aircraft heading changes, the gyroscope exerts a torque that balances the restoring torque from the spring at an angle proportional to the rate of change of heading.



(Instrument Flying Handbook FAA-H-8083-15)

If the gyroscope spins at 3,000 rpm, its moment of inertia is $1.7 \times 10^{-5}\,\mathrm{kg\,m^2}$, and the spring constant at the pointer is $10^{-3}\,\mathrm{N\,m\,rad^{-1}}$, find the angle through which the pointer turns if the aircraft makes one turn every two minutes. [6]

The situation is as for the spinning disc, except that in equilibrium the torque τ is provided by the spring, in such a way that the precession matches the rate of turn Ω of the aircraft. That is, *[1]*

$$\tau = \left|\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right| = \Omega L \sin\alpha \qquad [\,1\,]$$

where $\alpha \sim 90^\circ$ is the angle between the gyroscope axis and the vertical axis about which the aircraft turns. *[1]*

[In practical instruments, the gyroscope spin direction is chosen so that the gyroscope axis turns in the opposite direction to the aircraft, and hence remains roughly horizontal, when the aircraft is banked in the direction of turn.]

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If the pointer is displaced through an angle ϑ, and the spring exerts a restoring torque of $k\vartheta$, where $k = 10^{-3}\,\mathrm{N\,m\,rad^{-1}}$, then

$$\begin{aligned}\vartheta = \frac{\tau}{k} &= \frac{\Omega L \sin\alpha}{k} = \frac{\Omega I \omega \sin\alpha}{k} \\ &= \frac{(2\pi/120)\,\mathrm{s^{-1}} \times 1.7\times10^{-5}\,\mathrm{kg\,m^2} \times (2\pi \times 3\,000/60)\,\mathrm{rad\,s^{-1}} \times \sin 90^\circ}{10^{-3}\,\mathrm{N\,m\,rad^{-1}}} \\ &= 0.28\,\mathrm{rad} \equiv 16\,\mathrm{deg}.\end{aligned}$$

[1 mark per line] *[3]*

Explain how the instrument will be affected by rotation about (a) a lateral axis when the aircraft changes pitch and (b) a longitudinal axis when it rolls. [4]

(a) There will be no change in the instrument's indication [1] other than a slight change in the sensitivity to the rate of turn. Rotation about a lateral axis will merely rotate the instrument about the gyroscope axis, resulting in a small change in the relative speed of rotation that will quickly be corrected by the internal stabilization mechanisms. [1] *[2]*

(b) There will therefore be no change in the instrument's indication. [1] Rotation about a longitudinal axis will rotate the gimbal relative to the instrument; the restoring torque from the spring will cause a torque upon the gyroscope about a perpendicular axis, which will result in a reaction from the gyroscope bearings that will cause the gyroscope to precess about the longitudinal axis so as to reduce the spring torque to zero. [1] *[2]*

B4. A satellite orbits the Earth, subject only to the Earth's gravitational attraction. Explain why the vector angular momentum is conserved and why this means that the orbit lies in a plane. What other quantity or quantities is/are conserved? [5]

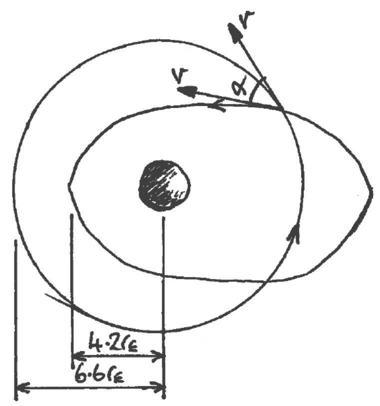
The gravitational attraction is a central force, so it exerts no torque about the Earth on the satellite [1]. Since the torque defines the rate of change of angular momentum, the angular momentum of the satellite about the Earth is constant [1]. *[2]*

Since the angular momentum $\mathbf{L}$ is conserved, and always perpendicular to the position $\mathbf{r}$ and its rate of change $\dot{\mathbf{r}} = \mathbf{p}/m$ [1], the satellite's motion remains in the plane defined by the initial $\mathbf{r}$ and $\mathbf{p}$ [1]. *[2]*

Since the gravitational attraction is a central, spherically symmetric force, the total energy of the satellite will be conserved. *[1]*

A spacecraft is moving in a circular, geostationary orbit of radius $6.6\,r_{\mathrm{E}}$ around the Earth, where r_{E} is the Earth's radius. A brief impulse from the spacecraft's rocket motors changes its direction of motion through an angle α towards the Earth, without any change in speed. At its closest approach, the new orbit is $4.2\,r_{\mathrm{E}}$ from the centre of the Earth.

Sketch the initial and final orbits, and indicate the spacecraft's velocities just before and after the rocket impulse. [2]



[2]

Find the angle α through which the spacecraft is deflected. [9]

We take v and r to be the spacecraft's initial speed and orbit radius, and V and R to be its speed and radial distance at the perigee of the elliptical orbit. The initial orbit gives us

$$\frac{mv^2}{r} = \frac{GMm}{r^2},$$ *[1]*

TURN OVER

while conservation of angular momentum requires

$$mv\cos\alpha r = mVR, \qquad [1]$$

hence

$$V = \frac{r}{R}\cos\alpha\, v. \qquad [1]$$

Conservation of energy similarly requires

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{2}mV^2 - \frac{GMm}{R}, \qquad [1]$$

hence

$$\begin{aligned}\frac{1}{2}v^2 - \frac{1}{2}\left(\frac{r}{R}\cos\alpha\right)^2 v^2 &= GM\left(\frac{1}{r}-\frac{1}{R}\right)\\ &= rv^2\left(\frac{1}{r}-\frac{1}{R}\right)\end{aligned}$$

so *[2]*

$$\cos\alpha = \frac{R}{r}\sqrt{1-2r\left(\frac{1}{r}-\frac{1}{R}\right)} = \frac{R}{r}\sqrt{\frac{2r}{R}-1} = 0.932 \qquad [2]$$

giving

$$\alpha = 21.3^\circ. \qquad [1]$$

Find the maximum distance of the spacecraft from the Earth's centre, and the period of the new orbit. [4]

The most direct way to solve this is by noting that the semimajor axis a is given by [1]

$$a = -\frac{GMm}{2E}$$

where the total energy E is the same for both orbits [1]. The semimajor axis is hence $2\times 6.6 r_{\mathrm{E}}$ so, since the closest approach is $4.2r_{\mathrm{E}}$, the farthest must be $(2\times 6.6-4.2)r_{\mathrm{E}} = 9.0r_{\mathrm{E}}$ [1]. Since, by Kepler's third law, the orbital period is proportional to the semimajor axis length$^{3/2}$, the period of the new orbit will be the same as that of the geostationary orbit: 1 day [1].

Without recalling this convenient property, we may determine from the working above,

$$\cos\alpha = \frac{R}{r}\sqrt{\frac{2r}{R}-1} \qquad [1]$$

Hence

$$\sin\alpha = \sqrt{1-\left(\frac{R}{r}\right)^2\left(\frac{2r}{R}-1\right)} = \sqrt{1-2\frac{R}{r}+\left(\frac{R}{r}\right)^2} = \sqrt{\left[1-\left(\frac{R}{r}\right)\right]^2} = \pm\left[1-\left(\frac{R}{r}\right)\right]$$

so

$$R = r\,(1 \mp \sin\alpha)\,. \qquad [\,1\,]$$

While we have derived this for the perigee radius R, which corresponds to the minus sign, it is also valid with the plus sign for the apogee, which hence has a radius

$$R = 6.6 r_{\mathrm{E}} \left(1 + \sin 21.3^{\circ}\right) = 9.0\, r_{\mathrm{E}}. \qquad [\,1\,]$$

Since, by Kepler's third law, the orbital period is proportional to the semimajor axis length$^{3/2}$, the period of the new orbit will be

$$\left(\frac{4.2 r_{\mathrm{E}} + 9.0 r_{\mathrm{E}}}{2 \times 6.6 r_{\mathrm{E}}}\right)^{\frac{3}{2}} \times 1\text{ day} = 1\text{ day!} \qquad [\,1\,]$$

[The Earth may be taken to be spherically symmetric, and its atmosphere and reduced mass effects may be neglected.]

END OF PAPER