SEMESTER 1 EXAMINATION 2006/07

WAVE PHYSICS

Duration: 120 MINS

*Answer **all** questions in **Section A** and two **and only two** questions in **Section B**.*

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it. Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

A Sheet of Physical Constants will be provided with this examination paper.

An outline marking scheme is shown in brackets to the right of each question.

Only university approved calculators may be used.

Section A

A1. A sinusoidal wave of frequency 40 Hz has a velocity of 120 m s^{-1}.

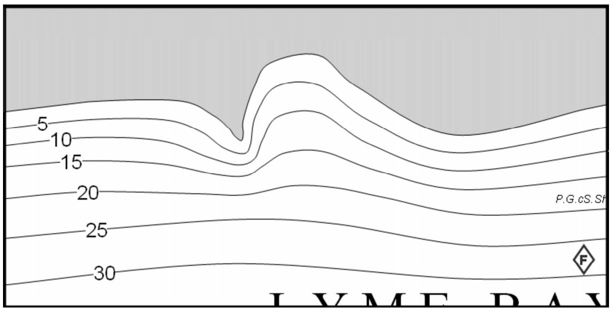
(a) Calculate the wave period, wavelength and angular frequency. [2]

(b) How far apart are two points whose displacements at any time differ in phase by 15°? [1]

(c) At a given point, what is the phase difference between two displacements occurring at times separated by 5 ms? [1]

A2. Explain briefly the process of *refraction*. [1]

The propagation speed v of long wavelength water waves is given in terms of the depth of water d and the acceleration due to gravity g by $v = \sqrt{gd}$. The roughly parallel wavefronts of ocean waves approach a shoreline whose gently-shelving sea-bed is indicated by the contour lines (showing depths in metres) in the figure below. Explain, illustrating your answer with a sketch, how the deep ocean waves will be affected as they approach the shore. [2]



Indicate on your sketch, with a brief explanation, a good spot to anchor your yacht for a tranquil Sunday lunch. [1]

A3. What is meant by *dispersion*? [1]

Give expressions for the *phase velocity* and *group velocity* of a wave whose angular frequency is given as a function of the wavenumber k by $\omega(k)$. [2]

Determine the relationship between the phase and group velocities for a quantum mechanical particle whose dispersion relation is

$$\omega = \frac{\hbar}{2m}k^2$$

where $\hbar$ is Planck's constant / 2π and m is the mass of the particle. [1]

A4. Why are *sinusoidal* wave motions so often considered? How are they related to *complex exponential* waves? [4]

A5. What is meant by wave *interference*. [1]

Describe how interference occurs in a *Michelson interferometer*, and outline how and for what such an instrument may be used. Illustrate your answer with a simple diagram. [3]

Section B

B1. Explain, with simple examples, what is meant by the *frequency spectrum* of a time-dependent wave signal $f(t)$. [4]

Describe the principles of *Fourier synthesis* and *Fourier analysis*, and indicate how and when they may be used. [4]

A radio signal, comprising a modulated carrier wave, may be written as $\mathrm{Re}[f(t)]$, where

$$f(t) = a_1\, e^{i\omega_1 t} + a_2\, e^{i\omega_2 t} + a_3\, e^{i\omega_3 t},$$

a_1, a_2, a_3, ω_1, ω_2 and ω_3 being constant parameters.

Show that the frequency spectrum of $f(t)$, given by its Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\,\mathrm{d}t$$

may be written as

$$F(\omega) = a_1\,\delta(\omega_1 - \omega) + a_2\,\delta(\omega_2 - \omega) + a_3\,\delta(\omega_3 - \omega)$$

where the Dirac δ-function is defined by

$$\delta(y) = \int_{-\infty}^{\infty} e^{iyz}\,\mathrm{d}z\,.$$ [2]

Hence determine the average frequency $\overline{\omega}$ of the *power* spectrum, defined by

$$\overline{\omega} = \frac{\int_0^{\infty} \omega\, F^*(\omega)\, F(\omega)\,\mathrm{d}\omega}{\int_0^{\infty} F^*(\omega)\, F(\omega)\,\mathrm{d}\omega}$$ [4]

Show that, if a_2 and a_3 are small in comparison with a_1 and the frequencies ω_1, ω_2 and ω_3 are similar, then $\overline{\omega} \approx \omega_1$. [2]

Show that the same result is obtained by calculating the *expectation value* of the *operator* $\hat{\omega} = -i(\mathrm{d}/\mathrm{d}t)$,

$$\langle\hat{\omega}\rangle = \frac{\int_{-\infty}^{\infty} f^*(t)\,\hat{\omega}\, f(t)\,\mathrm{d}t}{\int_{-\infty}^{\infty} f^*(t)\, f(t)\,\mathrm{d}t}\,.$$ [4]

It may be helpful to recall that, for any value of ω_0,

$$\int_{-\infty}^{\infty} \delta(\omega_0 - \omega)\,\mathrm{d}\omega = \int_{-\infty}^{\infty} \{\delta(\omega_0 - \omega)\}^2\;\mathrm{d}\omega = 1.$$

B2. Waves on a thin, flexible string of mass per unit length ρ and subject to a tension T are governed by the wave equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}$$

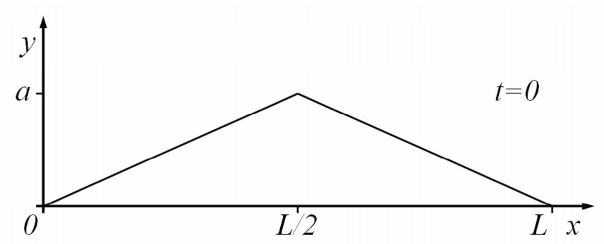
where the wave displacement at time t and position x is $y(x,t)$. Show that travelling waves of the form

$$y(x,t) = f(u)\,,$$

where $u = x - vt$, may be solutions to this wave equation, and find the two values of v which make them so. [4]

Hence write the general wave solution comprising an arbitrary superposition of these two solutions. [1]

The string of a harp is plucked by pulling it from its midpoint until it has the profile shown below and, at time $t = 0$, releasing the string from rest.



(a) What are the specific initial conditions, at time $t = 0$, which define the subsequent motion of the harp string? [2]

(b) Hence determine the motion of the harp string following its release from rest, and [4]

(c) illustrate your findings with a sketch showing the string displacement at time $t = L/8v$. [3]

[You may find it helpful to divide the string into four regions, bounded by the points $x = 0$, $x = (L/2) - vt$, $x = L/2$, $x = (L/2) + vt$, $x = L$.]

By considering the *boundary conditions* which apply where the string meets the frame of the harp at $x = 0$ and $x = L$, show that each of the two components of the string motion is reflected and inverted at the boundary. [2]

Hence determine and sketch the motion of the string at times $t = (L/4v)$, $2(L/4v)$, $3(L/4v)$ and $4(L/4v)$. [4]

B3. Explain what is meant by *diffraction*, and outline the conditions under which *Fraunhofer* diffraction may be observed. [5]

A naval towed-array sonar comprises a line of about 200 transducers, equally spaced every 1.5 m, that is towed behind a ship so that it lies in a straight line just below the surface of the water. An adjustable phase delay can be introduced electronically for each transducer, allowing the sonar beam to be steered without physically moving the array. The speed of sound in salt water may be taken to be around 1500 m s^{-1}.

(a) If the transducers are driven in phase at a constant frequency f, estimate the angular width of the (zeroth order) sonar beam. [3]

(b) A phase delay δ is now introduced between successive transducers. Determine how the angle θ through which the beam is steered depends upon δ. [2]

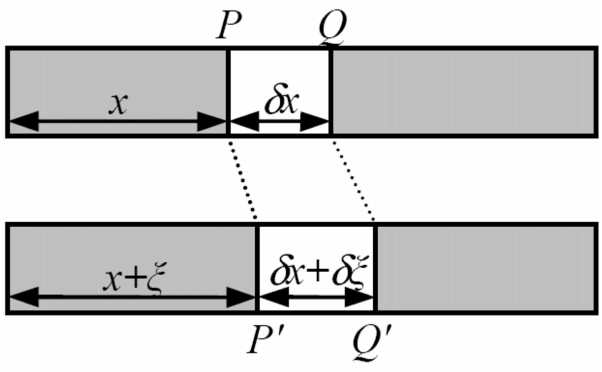
(c) For a given frequency f and phase delay δ, determine which 'diffraction orders' are possible, and hence find the maximum frequency that may be used if only one diffraction order is ever to be present as the beam is scanned from $\theta = -90^\circ$ to $\theta = 90^\circ$. [3]

(d) Hence determine the smallest angular width that can be obtained unambiguously (ie without generating extra beams) with such a system. [2]

(e) By writing the diffraction properties in terms of $\sin\theta$, determine the number of distinct beam directions that can be resolved within the scan range of $-1 \leq \sin\theta \leq 1$. [3]

If the phase delay δ can be given a different value for each transducer, suggest how δ should vary along the length of the array so as to produce a focused beam, rather than the parallel beam considered above. [2]

B4. The figure below shows the motion of a longitudinal volume element of an elastic medium which is displaced (bottom) by $\xi(x,t)$ from its rest position (top).



By considering the pressure within, and net force acting upon, the longitudinal element, and taking the density ρ and elasticity E of sea water to be 1026.4 kg m^{-3} and 2.32×10^9 Pa respectively, show that the wave equation governing its longitudinal motion will be

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 \xi}{\partial x^2}\,.$$ [5]

Hence estimate the speed of sound in water. [2]

The continuity conditions for the longitudinal displacement ξ of sound waves as they pass, at $x = x_0$, from a region A of density ρ_A and elasticity E_A into another region B of density ρ_A and elasticity E_A are

$$\begin{aligned} \xi_A(x_0,t) &= \xi_B(x_0,t) \\ E_A\frac{\partial \xi_A}{\partial x}(x_0,t) &= E_B\frac{\partial \xi_B}{\partial x}(x_0,t)\,. \end{aligned}$$

State the physical reason for each of these conditions. [2]

The displacements of travelling sinusoidal sound waves in the two regions A and B may be written in the form

$$\xi_{A,B}(x,t) = a_{A,B}\cos\left(\omega t - k_{A,B}(x-x_0)\right)$$

For a given angular frequency ω, state the possible values for the wavenumber $k_{A,B}$ in the two regions. [2]

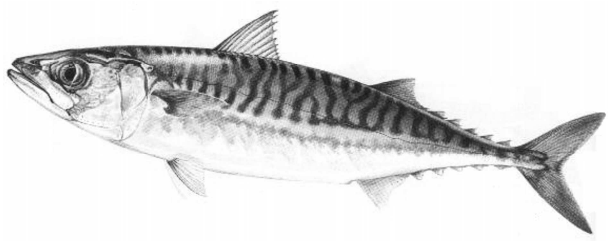
Hence show that when a sound wave of amplitude a_i passes from region A to region B it results in a reflected wave of amplitude a_r where

$$\frac{a_r}{a_i} = \frac{Z_A - Z_B}{Z_A + Z_B}$$

where the acoustic impedance Z is given in terms of the density ρ and elasticity E by $Z = \sqrt{E\rho}$. [5]

Given that the modulus of elasticity of the flesh of the Atlantic mackerel is around 2.58×10^9 Pa, and that the fish may be assumed homogeneous and neutrally buoyant, find the fraction of the wave *intensity* that is reflected when a fish-finder sonar pulse is normally incident upon the unsuspecting creature. [2]

What other considerations would determine the overall strength of sonar signal reflected by a shoal of Atlantic mackerel? [2]



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