WAVE PHYSICS examination Jan.'07.

A1. (a) wavelength $\lambda = \frac{v}{f} = \frac{120\,\mathrm{m\,s^{-1}}}{40\,\mathrm{s^{-1}}} = \underline{\underline{3\,\mathrm{m}.}}$ [1 mark]

period $\tau = \frac{1}{f} = \frac{1}{40\,\mathrm{s^{-1}}} = \underline{\underline{25\,\mathrm{ms}.}}$ [½ mark]

ang. freq. $\omega = 2\pi f = 2\pi . 40\,\mathrm{s^{-1}} = \underline{\underline{80\pi\ \mathrm{rads.s^{-1}}.}}$ [½ mark]

(b) $\frac{\Delta x}{\lambda} = \frac{\Delta\phi}{2\pi} \Rightarrow \Delta x = \frac{\Delta\phi}{2\pi}\lambda = \frac{15^\circ}{360^\circ}\,3\,\mathrm{m} = \underline{\underline{125\,\mathrm{mm}.}}$ [1 mark]

(c) $\frac{\Delta\phi}{2\pi} = \frac{\Delta t}{\tau} \Rightarrow \Delta\phi = 2\pi\frac{5\,\mathrm{ms}}{25\,\mathrm{ms}} = \underline{\underline{0.4\pi \equiv 72^\circ.}}$ [1 mark]

A2. Refraction is the process whereby the <u>direction</u> of propagation of a wave motion is changed as the wave moves between regions of different <u>speed</u> of propagation. [1 mark]

Since $v \propto \sqrt{h}$, the wave speed will decrease as the waves approach the shore. Where the coastline is straight, this causes plane waves to become more parallel to the shore (wavevector refracted towards the normal) so that they are nearly parallel by the time they break. [1 mark]
If there are promontories or coves, these act respectively like converging and diverging lenses, intensifying the waves around projections and reducing their strength in coves. [1 mark]

For the above reasons, coves or small bays will be the most comfortable anchorages. [1 mark]

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A3. Dispersion is the variation of wave propagation speed with the frequency of sinusoidal component, causing wavepackets to spread. [1 mark]

phase velocity $= \omega / k$ [1 mark]

group velocity $= \partial\omega / \partial k$ [1 mark]

For the quantum particle, $\omega = \frac{\hbar}{2m} k^2$

$\Rightarrow$ phase velocity $= \frac{\hbar}{2m} k$

group velocity $= \frac{\hbar}{2m} 2k$

$\Rightarrow$ group velocity is twice the phase velocity. [1 mark]

A4. Sinusoidal waves: answer should include some of the following:
- simplify analysis of differential equations (for linear systems)
- provide a complete basis set for the construction of any solution (")
- correspond to the emission from sources executing SHM or circular motion
- are the eigenmodes of dispersive systems
- are what we hear, musically, and see as colour
- are the basis set corresponding to standing waves (sep of variables)

[2 marks]

$$\exp i(kx - \omega t) \equiv \cos(kx - \omega t) + i\sin(kx - \omega t)$$

$$\cos(kx - \omega t) \equiv \mathrm{Re}\{\exp i(kx - \omega t)\}$$
$$\equiv \frac{1}{2}\{\exp i(kx - \omega t) + \exp -i(kx - \omega t)\}$$ [2 marks]

(or description in terms of superpositions with complex coefficients etc.)

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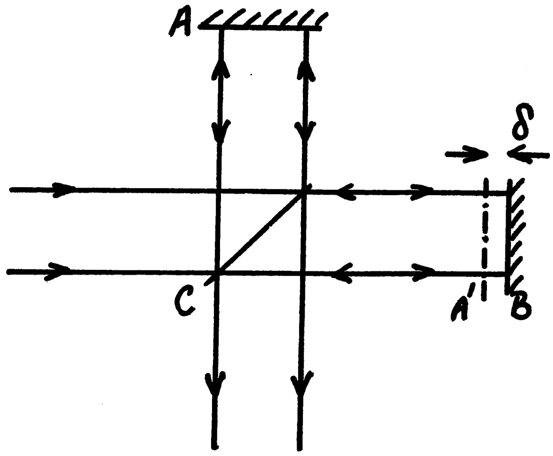
A5. Wave interference describes the addition of wave amplitudes (displacements) when two or more waves reach the same point at the same time. As the amplitude may be positive or negative, the waves may interfere constructively or destructively; the intensity is therefore not the simple sum of the component intensities. [1 mark]



[1 mark]

Incident light divided by semi-reflecting mirror C; reflected by mirror A or mirror B, before being recombined at C to pass to output. Depending upon path difference 2δ between two routes, the two corresponding contributions to the output will then interference, depending upon the frequencies present. [1 mark]

By recording the transmitted intensity as δ is scanned, we obtain information from which (by a Fourier transform) the spectrum of the incident light may be obtained. [1 mark]

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B1. The frequency spectrum indicates the respective strengths of components of various frequencies in a time-dependent signal or wave motion, where a single component will be a pure sine wave. [2 marks]

Examples might include the image recorded on the plate of a spectrograph, the musical notes on a stave, or the graphical output of a spectrum analyser. Single frequencies appear as δ-functions, harmonics as equally-spaced lines. [2 marks]

The principle of Fourier synthesis is that any function may be constructed by the judicious addition (superposition) of sinusoidal components, ie. [1 mark]

$$f(t) = \int_0^\infty a(\omega)\ \cos(\omega t + \phi(\omega))\ d\omega$$

The corresponding principle of Fourier analysis is that any function may be broken down into these component sinusoidal functions. [1 mark]

If we know how sinusoidal components behave in a particular system, we can hence determine the propagation of an arbitrary wave motion by breaking it into sinusoidal components, allowing for their known individual behaviour, and recombining them into the composite wave [1 mark]

Both the principles of Fourier synthesis and Fourier analysis depend, if they are to be so used, upon the system being linear. [1 mark]

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B1 cont'd. $f(t) = a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t} + a_3 e^{i\omega_3 t}$

$$\Rightarrow F(\omega) = \int_{-\infty}^{\infty} \left\{ a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t} + a_3 e^{i\omega_3 t} \right\} e^{-i\omega t}\, dt$$

$$= a_1 \int_{-\infty}^{\infty} e^{i(\omega_1-\omega)t}\, dt + a_2 \int_{-\infty}^{\infty} e^{i(\omega_2-\omega)t}\, dt + a_3 \int_{-\infty}^{\infty} e^{i(\omega_3-\omega)t}\, dt$$

$$= a_1\, \delta(\omega_1-\omega) + a_2\, \delta(\omega_2-\omega) + a_3\, \delta(\omega_3-\omega)$$

[2 marks]

where $\delta(y) = \int_{-\infty}^{\infty} e^{iyz}\, dz$.

$$\Rightarrow \int_0^{\infty} F^*(\omega) F(\omega)\, d\omega = \int_0^{\infty} \left\{ a_1^* \delta(\omega_1-\omega) + a_2^* \delta(\omega_2-\omega) + a_3^* \delta(\omega_3-\omega) \right\} \times \left\{ a_1 \delta(\omega_1-\omega) + a_2\, \delta(\omega_2-\omega) + a_3\, \delta(\omega_3-\omega) \right\} d\omega$$

$$= a_1^* a_1 \int_0^{\infty} \delta^2(\omega_1-\omega)\, d\omega + a_2^* a_2 \int_0^{\infty} \delta^2(\omega_2-\omega)\, d\omega + a_3^* a_3 \int_0^{\infty} \delta^2(\omega_3-\omega)\, d\omega$$

$$= a_1^* a_1 + a_2^* a_2 + a_3^* a_3$$

$$\int_0^{\infty} \omega F^*(\omega) F(\omega)\, d\omega = \int_0^{\infty} \omega \left\{ a_1^* \delta(\omega_1-\omega) + a_2^* \delta(\omega_2-\omega) + a_3^* \delta(\omega_3-\omega) \right\} \times \left\{ a_1 \delta(\omega_1-\omega) + a_2 \delta(\omega_2-\omega) + a_3 \delta(\omega_3-\omega) \right\} d\omega$$

$$= \omega_1 a_1^* a_1 \int_0^{\infty} \delta^2(\omega_1-\omega)\, d\omega + \omega_2 a_2^* a_2 \int_0^{\infty} \delta^2(\omega_2-\omega)\, d\omega + \omega_3 a_3^* a_3 \int_0^{\infty} \delta^2(\omega_3-\omega)\, d\omega$$

$$= \omega_1 a_1^* a_1 + \omega_2 a_2^* a_2 + \omega_3 a_3^* a_3$$

$$\Rightarrow \bar{\omega} = \frac{\omega_1 a_1^* a_1 + \omega_2 a_2^* a_2 + \omega_3 a_3^* a_3}{a_1^* a_1 + a_2^* a_2 + a_3^* a_3}.$$

[4 marks]

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B1 cont'd. If $a_{2,3} \ll a_1$, then $(a_2^* a_2), (a_3^* a_3) \ll (a_1^* a_1)$ so, if $\omega_1 \approx \omega_2 \approx \omega_3$,

$$\bar{\omega} \approx \frac{\omega_1 a_1^* a_1}{a_1^* a_1} = \omega_1 .$$

[2 marks]

$$\int_{-\infty}^{\infty} f^*(t) f(t)\, dt = \int_{-\infty}^{\infty} \left(a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t} + a_3 e^{i\omega_3 t}\right)^* \left(a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t} + a_3 e^{i\omega_3 t}\right) dt$$

$$= \int_{-\infty}^{\infty} a_1^* a_1 + a_2^* a_2 + a_3^* a_3 + \text{oscillating terms } dt$$

$$= \left(a_1^* a_1 + a_2^* a_2 + a_3^* a_3\right) \int_{-\infty}^{\infty} dt$$

$$\int_{-\infty}^{\infty} f^*(t)\, \hat{\omega}\, f(t)\, dt = \int_{-\infty}^{\infty} \left(a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t} + a_3 e^{i\omega_3 t}\right)^* (-i) \frac{d}{dt} \left(a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t} + a_3 e^{i\omega_3 t}\right) dt$$

$$= -i \int_{-\infty}^{\infty} \left(a_1^* e^{-i\omega_1 t} + a_2^* e^{-i\omega_2 t} + a_3^* e^{-i\omega_3 t}\right)\left(i\omega_1 a_1 e^{i\omega_1 t} + i\omega_2 a_2 e^{i\omega_2 t} + i\omega_3 a_3 e^{i\omega_3 t}\right) dt$$

$$= \int_{-\infty}^{\infty} a_1^* a_1 \omega_1 + a_2^* a_2 \omega_2 + a_3^* a_3 \omega_3 + \text{oscillating terms } dt$$

$$= \left(a_1^* a_1 \omega_1 + a_2^* a_2 \omega_2 + a_3^* a_3 \omega_3\right) \int_{-\infty}^{\infty} dt$$

$$\Rightarrow \langle \hat{\omega} \rangle = \frac{\omega_1 a_1^* a_1 + \omega_2 a_2^* a_2 + \omega_3 a_3^* a_3}{a_1^* a_1 + a_2^* a_2 + a_3^* a_3} \equiv \bar{\omega} .$$

[4 marks]

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B2. Substitute the trial form into the wave equation given, using the results

$$\frac{\partial}{\partial t} = \frac{\partial u}{\partial t}\frac{d}{du} \; ; \; \frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{d}{du}$$

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = \left(\frac{\partial u}{\partial t}\right)^2 \frac{d^2 y}{du^2} = v^2 \frac{d^2 y}{du^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \left(\frac{\partial u}{\partial x}\right)^2 \frac{d^2 y}{du^2} = \frac{d^2 y}{du^2}$$

[2 marks]

so, substituting into the wave equation,

$$v^2 \frac{d^2 y}{du^2} = \frac{T}{\rho}\frac{d^2 y}{du^2}$$

[1 mark]

i.e. the trial form is a solution provided that $v^2 = T/\rho$

$$\therefore \; v = \pm\sqrt{T/\rho}.$$

[1 mark.]

So the general solution will be

$$y(x,t) = f(x-vt) + g(x+vt) \quad \text{where } v = |\sqrt{T/\rho}|.$$

[1 mark]

(a) at $t=0$, $y(x,0)$ is as shown $\left(y(x<\tfrac{L}{2},0) = \frac{2a}{L}x \; ; \; y(x>\tfrac{L}{2},0) = \frac{2a}{L}(L-x)\right)$ [1 mark]

$\frac{\partial y}{\partial t}(x,0) = 0$ i.e. string initially at rest. [1 mark]

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B2 cont'd (b) If $y(x,t) = f(x-vt) + g(x+vt)$
and we denote the sketched form $y_0(x)$, then

$$y_0(x) = f(x) + g(x)$$ [1 mark]

and $$0 = -vf'(x,0) + vg'(x,0) \quad \text{where } f' \equiv \frac{df}{du} \text{ etc.}$$ [1 mark]

$$= -v\left\{\frac{\partial f}{\partial x} - \frac{\partial g}{\partial x}\right\}$$

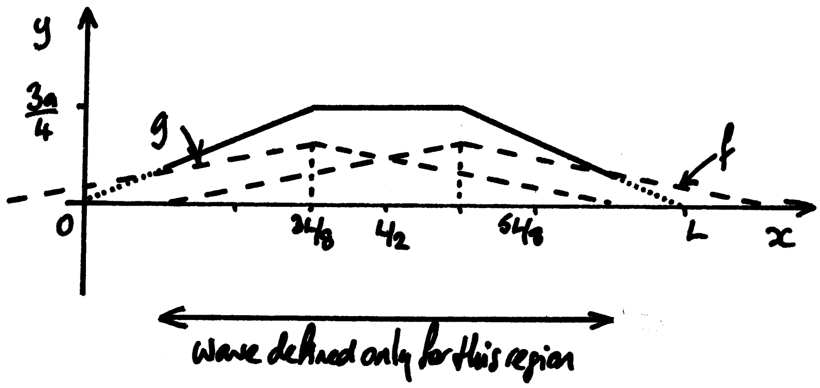
$\Rightarrow$ $$f(x) = g(x) + \text{const.}$$ [1 mark]

and we may set the constant to zero.

$$\Rightarrow \quad f(x) = g(x) = \frac{1}{2}y_0(x)$$

so $$\underline{\underline{y(x,t) = \frac{1}{2}y_0(x-vt) + \frac{1}{2}y_0(x+vt).}}$$ [1 mark]

(c) At $t = \frac{L}{8v}$, $vt = \frac{L}{8}$.



[3 marks]

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B2 cont'd Boundary conditions: $y(0,t) = y(L,t) = 0$

so, if $y(x,t) = f(x-vt) + g(x+vt)$,

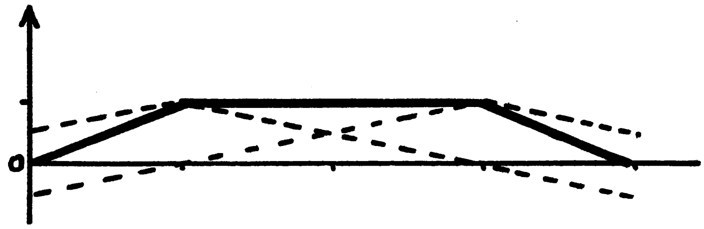
$$f(-vt) + g(vt) = 0$$
$$f(L-vt) + g(L+vt) = 0$$

ie. $$f(0,t) = -g(0,t)$$
$$f(L,t) = -g(L,t)$$

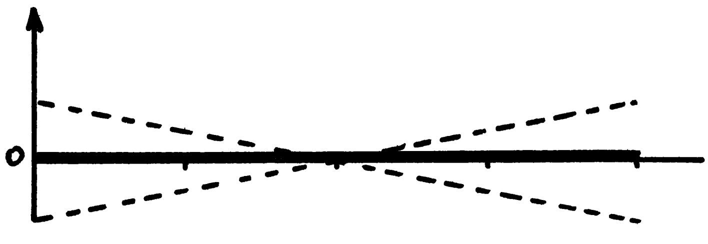
so each component is reflected and inverted at the boundary. [2 marks]

Hence f and g are defined beyond the initial region, and we may determine the general subsequent motion: (only string needs to be shown)

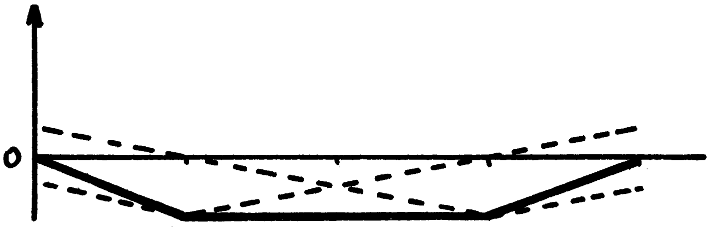
$t = \frac{L}{4v}$



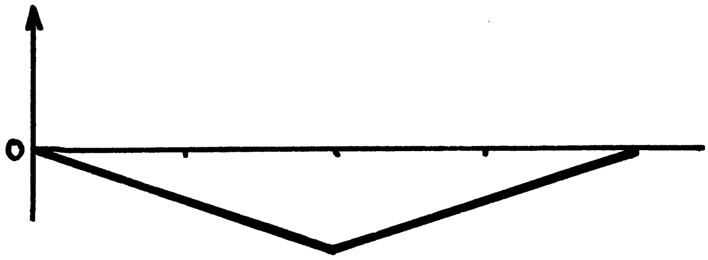
$t = 2\frac{L}{4v}$



$t = 3\frac{L}{4v}$



$t = 4\frac{L}{4v}$



[4 marks]

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B3. Diffraction is the interference that results from the spatially-dependent partial destruction or retardation of a wavefront. [3 marks]

Fraunhofer diffraction is that which is observed in the image plane of the source, whereby the path length or accrued phase for paths through a given point in the diffracting object is a linear function of the coordinates of that point. For plane wave illumination of a diffracting object, Fraunhofer diffraction is observed at distances $\gg$ object dimensions, or in the focal plane of a lens used to focus the transmitted wave. [2 marks]

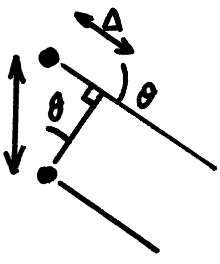
(a) angular width $\Delta\theta \sim \frac{\lambda}{L}$ where array length $L \sim 200 \times 1.5\,\mathrm{m} = 300\,\mathrm{m}$

$$= \frac{v/f}{L} \quad \text{where } v \sim 1500\,\mathrm{m\,s^{-1}}$$

$$= \frac{1500}{300}\frac{1}{f} = \underline{\underline{5/f \text{ radians.}}}$$ [3 marks]

(most definitions of 'width' will be within a factor of 2.)

(b)



$$\sin\theta = \frac{\Delta}{d} = \frac{\delta}{2\pi}\frac{\lambda}{d}$$

$$\Rightarrow \theta = \sin^{-1}\frac{\lambda}{2\pi d}\delta = \underline{\underline{\sin^{-1}\left(\frac{v}{2\pi f d}\delta\right)}}$$ [2 marks]

$$= \sin^{-1}\left(\frac{1500}{f\ 1.5}\frac{\delta}{2\pi}\right) = \sin^{-1}\left(\frac{1000}{f}\frac{\delta}{2\pi}\right).$$

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B3 cont'd (c) For a diffraction order,

$$d\sin\theta + \frac{\delta}{2\pi}\lambda = n\lambda$$

$$\Rightarrow \underline{\underline{\sin\theta = \left(n - \frac{\delta}{2\pi}\right)\frac{\lambda}{d} = \left(n - \frac{\delta}{2\pi}\right)\frac{v}{fd}.}}$$ [1 mark]

Since $-1 \leq \sin\theta \leq +1$, we need $\Delta\sin\theta \geq 2$ for adjacent orders ($\Delta n = 1$)

where $\Delta\sin\theta = \Delta\left(n - \frac{\delta}{2\pi}\right)\frac{\lambda}{d}$

$= \lambda/d = \frac{v}{fd} \geq 2$ for a single beam

$$\Rightarrow f \leq \frac{v}{2d} = \frac{1500\,\mathrm{m\,s^{-1}}}{2 \times 1.5\,\mathrm{m}} \underline{\underline{= 500\,\mathrm{Hz}.}}$$ [2 marks]

(d) Smallest value of $\Delta\theta \sim 5/f$ will be obtained with max. frequency

ie. $\Delta\theta \sim \frac{5}{500} \underline{\underline{= 0.01^c \equiv 0.57^\circ.}}$ [2 marks]

(e) Resolution more precisely defined as $\Delta(L\sin\theta) = \lambda$ ie. $\Delta\sin\theta = \frac{\lambda}{L}$ [2 marks]

$\Rightarrow$ to fill $-1 \to +1$ in steps of 0.01 gives 200 beams. [1 mark]

For a lens, we need a retardation that is greater at the centre of the array than at the ends. [1 mark]

If the beam is ~ perpendicular to the array, the phase should depend quadratically upon distance from the centre

ie. $\delta_n = \alpha\left(n - \frac{N}{2}\right)^2.$ [1 mark]

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B4. Pressure in element (wrt ambient) $= -E \dfrac{(\xi_Q - \xi_P)}{\delta x}$ [1 mark]

so, taking the limit $\delta x \to 0$, $\quad P(x) = -E \dfrac{\partial \xi}{\partial x}$. [1 mark]

Mass of element $= \rho\, \delta x\, A$ where A = element cross-sectional area

Force on element $= \{P(x) - P(x+\delta x)\} A$ [1 mark]

$\Rightarrow \rho A \delta x \dfrac{\partial^2 \xi}{\partial t^2} = -A\{P(x+\delta x) - P(x)\}$ [1 mark]

Dividing by δx and again taking the limit $\delta x \to 0$,

$$\rho A \frac{\partial^2 \xi}{\partial t^2} = -A \frac{\partial P}{\partial x}$$

$$= -A \frac{\partial}{\partial x}\left(-E \frac{\partial \xi}{\partial x}\right)$$

$$\Rightarrow \underline{\underline{\frac{\partial^2 \xi}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 \xi}{\partial x^2}}}.$$ [1 mark]

By substitution of $y = f(x - vt)$ or otherwise, we identify the wave speed with

$$v = \sqrt{E/\rho}$$

so, with $E = 2.32 \times 10^9$ Pa, $\rho = 1026.4$ kg.m^{-3},

$$\underline{\underline{v = 1503 \text{ m.s}^{-1}}}.$$ [2 marks]

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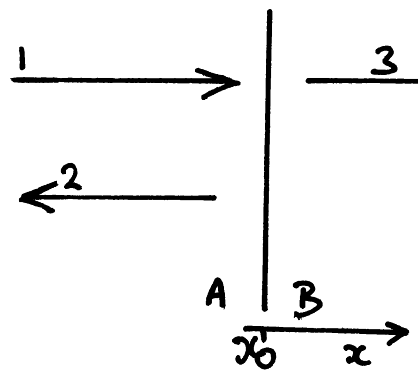
B4 cont'd. $\mathcal{E}_A = \mathcal{E}_B$ so that there are no voids within the medium [1 mark]

$$E_A \frac{\partial \mathcal{E}_A}{\partial x} = E_B \frac{\partial \mathcal{E}_B}{\partial x}$$ ie. $P_A = P_B$ so that the force falls to zero as the volume of any element is reduced to zero – ie. finite acceleration. [1 mark]

By substitution or otherwise,

$$\left(\frac{\omega}{k}\right)^2 = \frac{E}{\rho}$$

$$\Rightarrow \quad \underline{\underline{k = \pm\omega\sqrt{\frac{\rho}{E}}}}. \quad \text{ie. } \underline{\underline{k_{A,B} = \pm\omega\sqrt{\frac{\rho_{A,B}}{E_{A,B}}}}}$$ [2 marks]



let waves be

1 $\quad a_i \cos(\omega t - k_A(x - x_0))$ where $k_A = +\omega\sqrt{\frac{\rho_A}{E_A}}$

2 $\quad a_r \cos(\omega t + k_A(x - x_0))$

3 $\quad a_t \cos(\omega t - k_B(x - x_0))$ [1 mark]

$$\Rightarrow \mathcal{E}_A(x,t) = a_i \cos(\omega t - k_A(x-x_0)) + a_r \cos(\omega t - k_A(x-x_0))$$

$$\mathcal{E}_B(x,t) = a_t \cos(\omega t - k_B(x-x_0))$$ [1 mark]

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B4 cont'd. Applying the boundary conditions at $x = x_0$,

$$a_i \cos\omega t + a_r \cos\omega t = a_t \cos\omega t$$

$$\Rightarrow \quad a_i + a_r = a_t$$ [1 mark]

and $E_A a_i k_A \sin\omega t - E_A a_r k_A \sin\omega t = E_B a_t k_B \sin\omega t$

$$\Rightarrow \quad E_A k_A (a_i - a_r) = E_B k_B a_t$$ [1 mark]

Substituting $a_t = a_i + a_r$ now gives

$$E_A k_A (a_i - a_r) = E_B k_B (a_i + a_r)$$

$$\Rightarrow \quad a_i (E_A k_A - E_B k_B) = a_r (E_A k_A + E_B k_B)$$

$$\Rightarrow \quad \frac{a_r}{a_i} = \frac{E_A k_A - E_B k_B}{E_A k_A + E_B k_B} = \underline{\underline{\frac{Z_A - Z_B}{Z_A + Z_B}}} \quad \text{where } Z = \sqrt{E\rho}.$$ [1 mark]

Fish neutrally buoyant so $\rho_{FISH} = \rho_{WATER}$

$$\Rightarrow \quad \left|\frac{a_r}{a_i}\right|^2 = \left|\frac{\sqrt{E_F} - \sqrt{E_W}}{\sqrt{E_F} + \sqrt{E_W}}\right|^2 \quad \text{where } E_F = 2.58 \times 10^9 \text{ Pa}, \quad E_W = 2.32 \times 10^9 \text{ Pa}$$

$$\Rightarrow \text{intensity reflectivity} = \left|\frac{\sqrt{2.58} - \sqrt{2.32}}{\sqrt{2.58} + \sqrt{2.32}}\right|^2 = \underline{\underline{0.027^2 = 7 \times 10^{-4}}}.$$ [2 marks]

Other factors: number of fish in shoal
size of shoal
size of fish (diffraction)
shape of fish (scattering angles)
thickness of fish (double reflection)
homogeneity

[2 marks for something sensible]