A1. (a) Wavelength $\lambda = \frac{v}{f} = \frac{120\text{m/s}}{40\text{ s}^{-1}} = 3\text{ m}$. [1 mark]

Period $T = \frac{1}{f} = \frac{1}{40\text{ s}^{-1}} = 25\text{ ms}$. [\frac{1}{2}\text{ mark}]

Angular freq. $\omega = 2\pi f = 2\pi \times 40\text{ s}^{-1} = 80\pi\text{ rad/s}$. [\frac{1}{2}\text{ mark}]

(b) $\frac{\Delta x}{\lambda} = \frac{\Delta \phi}{2\pi} \Rightarrow \Delta x = \frac{\Delta \phi \lambda}{2\pi} = \frac{15^\circ}{360^\circ} \times 3\text{ m} = 125\text{ mm}$. [1 mark]

(c) $\frac{\Delta \phi}{2\pi} = \frac{\Delta t}{T} \Rightarrow \Delta \phi = \frac{2\pi \times 5\text{ ms}}{2\text{ ms}} = 0.4\pi \approx 72^\circ$. [1 mark]

A2. Refraction is the process whereby the direction of propagation of a wave motion is changed as the wave moves between regions of different speed of propagation. [1 mark]

Since $v \propto \frac{1}{\sqrt{n}}$, the wave speed will decrease as the waves approach the shore. Where the coastline is straight, this causes plane waves to become more parallel to the shore (wave vector refracted towards the normal) so that they are nearly parallel by the time they break. [1 mark]

If there are promontories or cones, these act respectively like converging and diverging lenses, intensifying the waves around projections and reducing their strength in cones. [1 mark]

For the above reasons, cones or small bays will be the most susceptible anchorage. [1 mark]
A3. Dispersion is the variation of wave propagation speed with the frequency of sinusoidal component, causing wave packets to spread.

\[ \text{phase velocity} = \frac{\omega}{k} \]  
\[ \text{group velocity} = \frac{\partial \omega}{\partial k} \]  

For the quantum particle, \( \omega = \frac{\hbar}{2m} k^2 \)  

\[ \Rightarrow \quad \text{phase velocity} = \frac{\hbar}{2m} k \]  
\[ \Rightarrow \quad \text{group velocity} = \frac{\hbar}{2m} 2k \]  

\[ \Rightarrow \quad \text{group velocity is twice the phase velocity.} \]  

Alt. Sinusoidal waves: answer should include some of the following:
- Suggestion analysis of differential equations (linear systems)
- Provide a complete basis set for the construction of any solution (\( n \))
- Ensure that the emission from sources executing simple harmonic motion
- Are these eigenmodes of dispersive systems
- Are what we hear, musically, and see as colours
- Are the basis set corresponding to standing waves (separation variables)

\[ \exp i (kx-\omega t) \equiv \cos( kx - \omega t ) + i \sin( kx - \omega t ) \]  
\[ \cos (kx-\omega t) \equiv \Re \{ \exp i (kx-\omega t) \} \]  
\[ \equiv \frac{1}{2} \{ \exp i (kx-\omega t) + \exp -i (kx-\omega t) \} \]  

(or description in terms of superposition with complex coefficients etc.)
A5. Wave interference describes the addition of wave amplitudes (displacements) when two or more waves reach the same point at the same time. As the amplitude may be positive or negative, the waves may interfere constructively or destructively; the intensity is therefore not the simple sum of the component intensities.

[1 mark]

[Diagram]

Incident light divided by semi-reflecting mirror C, reflected by mirror A or mirror B, before being recombined at C to pass to output. Depending upon path difference \( \delta \) between two routes, the two corresponding contributions to the output will then interfere, depending upon the frequencies present.

By scanning the transmitted intensity as \( \delta \) is scanned, we obtain information from which (by a Fourier transform) the spectrum of the incident light may be obtained.

[1 mark]
B1. The frequency spectrum indicates the respective strengths of components of various frequencies in a time-dependent signal or wave motion, where a single component will be a pure sine wave. [2 marks]

Examples might include the image recorded on the plate of a spectograph, the musical notes on a stave, or the graphical output of a spectrum analyser. Single frequencies appear as δ-functions, harmonics as equally-spaced lines. [2 marks]

The principle of Fourier synthesis is that any function may be constructed by the judicious addition (superposition) of sinusoidal components, i.e. [1 mark]

\[ f(t) = \int a(\omega) \cos(\omega t + \phi(\omega)) \, d\omega \]

The corresponding principle of Fourier analysis is that any function may be broken down into these constituent sinusoidal functions. [1 mark]

If we know how sinusoidal components behave in a particular system, we can hence determine the propagation of an arbitrary wave motion by breaking it into sinusoidal components, allowing for their known individual behaviour, and recombining them into the composite wave. [1 mark]

Both the principles of Fourier synthesis and Fourier analysis depend, if they are to be so used, upon the system being linear. [1 mark]
\[ f(t) = a_1 e^{i\omega t} + a_2 e^{i\omega_2 t} + a_3 e^{i\omega_3 t} \]

\[ \Rightarrow F(\omega) = \int_{-\infty}^{\infty} \left\{ a_1 e^{i\omega t} + a_2 e^{i\omega_2 t} + a_3 e^{i\omega_3 t} \right\} e^{-i\omega t} dt \]

\[ = a_1 \int_{-\infty}^{\infty} \delta(\omega_1 - \omega) dt + a_2 \int_{-\infty}^{\infty} \delta(\omega_2 - \omega) dt + a_3 \int_{-\infty}^{\infty} \delta(\omega_3 - \omega) dt \]

\[ = a_1 \delta(\omega - \omega_1) + a_2 \delta(\omega - \omega_2) + a_3 \delta(\omega - \omega_3) \quad [2 \text{ marks}] \]

where \( \delta(y) = \int_{-\infty}^{\infty} e^{iyz} dz \).

\[ \Rightarrow \int_{-\infty}^{\infty} F^*(\omega) F(\omega) d\omega = \int_{-\infty}^{\infty} \left\{ a_1^* \delta(\omega_1 - \omega) + a_2^* \delta(\omega_2 - \omega) + a_3^* \delta(\omega_3 - \omega) \right\} \times \left\{ a_1 \delta(\omega_1 - \omega) + a_2 \delta(\omega_2 - \omega) + a_3 \delta(\omega_3 - \omega) \right\} d\omega \]

\[ = a_1^* a_1 \int_{-\infty}^{\infty} \delta^2(\omega_1 - \omega) d\omega + a_2^* a_2 \int_{-\infty}^{\infty} \delta^2(\omega_2 - \omega) d\omega + a_3^* a_3 \int_{-\infty}^{\infty} \delta^2(\omega_3 - \omega) d\omega \]

\[ = \omega_1 a_1^* a_1 + \omega_2 a_2^* a_2 + \omega_3 a_3^* a_3 \]

\[ \Rightarrow \bar{\omega} = \frac{\omega_1 a_1^* a_1 + \omega_2 a_2^* a_2 + \omega_3 a_3^* a_3}{a_1^* a_1 + a_2^* a_2 + a_3^* a_3} \quad [4 \text{ marks}] \]
Blockd. If \( a_{23} \ll a_1 \), then \((a_{1}^* a_{1}) (a_{2}^* a_{2}) (a_{3}^* a_{3}) \ll (a_{1}^* a_{1})\) so, if \( \omega_1 \approx \omega_2 \approx \omega_3 \),

\[
\bar{\omega} \propto \frac{\omega_1 a_{1}^* a_{1}}{a_{1}^* a_{1}} = \omega_1. \quad [2 \text{ marks}]
\]

\[
\int f^*(t) k(t) \, dt = \int (a_1 e^{i t \omega_1} + a_2 e^{i t \omega_2} + a_3 e^{i t \omega_3}) (a_1^* e^{-i t \omega_1} + a_2^* e^{-i t \omega_2} + a_3^* e^{-i t \omega_3}) \, dt
\]

\[
= \int a_1^* a_1 + a_2^* a_2 + a_3^* a_3 + \text{oscillatory terms} \, dt
\]

\[
= (a_1^* a_1 + a_2^* a_2 + a_3^* a_3) \int dt
\]

\[
\int f^*(t) \hat{\nu}(t) \, dt = \int (a_1 e^{i t \omega_1} + a_2 e^{i t \omega_2} + a_3 e^{i t \omega_3}) (i \hat{\nu}_1 e^{-i t \omega_1} + i \hat{\nu}_2 e^{-i t \omega_2} + i \hat{\nu}_3 e^{-i t \omega_3}) \, dt
\]

\[
= -i \int (a_1 e^{i t \omega_1} + a_2 e^{i t \omega_2} + a_3 e^{i t \omega_3}) (i a_1^* e^{-i t \omega_1} + i a_2^* e^{-i t \omega_2} + i a_3^* e^{-i t \omega_3}) \, dt
\]

\[
= \int a_1^* a_1 \omega_1 + a_2^* a_2 (\omega_2 - \omega_1) + a_3^* a_3 (\omega_3 - \omega_1) + \text{oscillatory terms} \, dt
\]

\[
= (a_1^* a_1 \omega_1 + a_2^* a_2 (\omega_2 - \omega_1) + a_3^* a_3 (\omega_3 - \omega_1)) \int dt
\]

\[
\Rightarrow \langle \hat{\omega} \rangle = \frac{\omega_1 a_{1}^* a_{1} + \omega_2 a_{2}^* a_{2} + \omega_3 a_{3}^* a_{3}}{a_{1}^* a_{1} + a_{2}^* a_{2} + a_{3}^* a_{3}} = \bar{\omega}. \quad [4 \text{ marks}]
\]
B2. Substitute the trial form into the wave equation given, using the results
\[ \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} \]
\[ \frac{\partial}{\partial t} = \frac{\partial u}{\partial t} \frac{\partial}{\partial u} \]

\[ \Rightarrow \frac{\partial^2 y}{\partial t^2} = (\frac{\partial u}{\partial t})^2 \frac{\partial^2 y}{\partial u^2} = v^2 \frac{\partial^2 y}{\partial u^2} \]
\[ \frac{\partial^2 y}{\partial x^2} = (\frac{\partial u}{\partial x})^2 \frac{\partial^2 y}{\partial u^2} = \frac{\partial^2 y}{\partial u^2} \]

[2 marks]

so, substituting into the wave equation,
\[ v^2 \frac{\partial^2 y}{\partial u^2} = \frac{T}{\rho} \frac{\partial^2 y}{\partial u^2} \]

[1 mark]

\textbf{v}. The trial form is a solution provided that \( v^2 = \frac{T}{\rho} \)

\[ \therefore v = \pm \sqrt{\frac{T}{\rho}}. \]

[1 mark]

So the general solution will be
\[ y(x,t) = f(x-vt) + g(x+vt) \quad \text{where } v = \sqrt{\frac{T}{\rho}}. \]

[1 mark]

(a) at \( t = 0 \), \( y(x,0) \) is as shown \((y(x,0) = \begin{cases} \frac{2a}{L} & x \leq \frac{L}{2} \\ \frac{2a}{L} & x > \frac{L}{2} \end{cases}\))

\[ \frac{\partial y}{\partial t}(x,0) = 0 \quad \text{i.e. string initially at rest}. \]

[1 mark] [1 mark]
B2 cont (b) \( y(x,t) = f(x-vt) + g(x+vt) \)

and we denote the skewed form \( y_0(x) \), then

\[
y_0(x) = f(x) + g(x)
\]

and \( \dot{0} = -vf'(x_0) + vg'(x_0) \) where \( f' = \frac{df}{dx} \) etc.

\[
\dot{0} = -v \{ \frac{df}{dx} - \frac{dg}{dx} \}
\]

\[
\Rightarrow f(x) = g(x) + \text{const.}
\]

and we may set the constant to zero.

\[
\Rightarrow f(x) = g(x) = \frac{1}{2} y_0(x)
\]

so \( y(x,t) = \frac{1}{2} y_0(x-vt) + \frac{1}{2} y_0(x+vt) \).

(c) At \( t = \frac{L}{8v} \), \( vt = \frac{L}{8} \).

(c) At \( t = \frac{L}{8v} \), \( vt = \frac{L}{8} \).

[3 marks]
Boundary conditions: \( y(0,t) = y(L,t) = 0 \)

so, if \( y(x,t) = f(x-vt) + g(x+vt) \),

\[
\begin{align*}
f(-vt) + g(vt) &= 0 \\
f(L-vt) + g(L+vt) &= 0
\end{align*}
\]

ie. \( f(0,t) = -g(0,t) \)
\( f(L,t) = -g(L,t) \)

so each component is reflected and inverted at the boundary. [2 marks]

Hence \( f \) and \( g \) are defined beyond the initial region, and we may determine the general subsequent motion (only string needs to be shown) [4 marks]
B3. Diffraction is the interference that results from the spatially-dependent periodic destruction or retardation of a wavefront. [2 marks]

Fraunhofer diffraction is that which is observed in the image plane of the source, where the path length or accrued phase for paths through a given point in the diffracting object is a linear function of the coordinates of that point. For plane wave illumination of a diffracting object, Fraunhofer diffraction is observed at distances \( \gg \) object dimension, or in the back plane of a lens used to focus the transmitted wave. [2 marks]

(a) Angular width \( \Delta \theta \sim \frac{\lambda}{L} \) where array length \( L \sim 200 \times 1.5 \text{m} = 300 \text{m} \)

\[
\Delta \theta = \frac{v/f}{L} \quad \text{where} \quad v \sim 1500 \text{m/s}
\]

\[
= \frac{1500}{200} \frac{1}{f} = \frac{5}{f} \text{ radians.}
\]

(Most definitions of width will be within a factor of 2.) [3 marks]

(b) \[
\sin \theta = \frac{\Delta}{2 \pi d} = \frac{\lambda}{2 \pi d} \]

\[
\Rightarrow \theta = \sin^{-1} \left( \frac{\lambda}{2 \pi d} \right) = \sin^{-1} \left( \frac{v}{2 \pi f d} \right) = \sin^{-1} \left( \frac{1500}{1.5 \cdot \frac{1}{f} \text{ m/s}} \frac{\lambda}{2 \pi} \right) = \sin^{-1} \left( \frac{1000 \lambda}{2 \pi} \right). \] [2 marks]
B3(c) For a diffraction order, 
\[ d \sin \theta + \frac{d}{2\pi} \lambda = n\lambda \]
\[ \Rightarrow \sin \theta = \frac{(n-\frac{d}{2\pi}) \lambda}{d} = \frac{(n-\frac{d}{2\pi}) v}{fd}. \]  

[1 mark]

Since \(-1 \leq \sin \theta \leq 1\), we need \(\Delta \sin \theta \geq 2\) for adjacent orders (\(n=\ldots\))

where \(\Delta \sin \theta = \Delta \left(\frac{d}{2\pi} \frac{\lambda}{d}\right)\)
\[ = \frac{\lambda}{d} \frac{v}{fd} = 2 \text{ for a single beam} \]
\[ \Rightarrow f \leq \frac{v}{2d} = \frac{1500 \text{ m/s}}{2 \times 1.5 \text{ m}} = 500 \text{ Hz}. \]  

[2 marks]

(d) Smallest value of \(\Delta \theta \sim \frac{5}{f}\) will be obtained with max frequency
\[ \Rightarrow \Delta \theta \sim \frac{5}{500} = 0.01^\circ \equiv 0.57^\circ. \]  

[2 marks]

(e) Resolution more precisely defined as \(\Delta \sin \theta = \lambda\) \(\Leftrightarrow \Delta \sin \theta = \frac{\lambda}{d}\)  
\[ \Rightarrow \text{to fill } -1 \rightarrow 1 \text{ in steps of } 0.01 \text{ gives } 200 \text{ beams.} \]  

[1 mark]

For a lens, we need a retardation that is greater at the centre of the array than at the ends.

If the beam is perpendicular to the array, the phase should depend quadratically upon distance from the centre
\[ \Rightarrow \delta_n = \alpha (n-\frac{1}{2})^2. \]  

[1 mark]
B4. Pressure in element = $-E \frac{\Delta \varepsilon}{\Delta x}$

so, taking the limit $\Delta x \to 0$, $P(x) = -E \frac{\partial \varepsilon}{\partial x}$. [1 mark]

Mass of element = $\rho \Delta x \ A$ where $A = \text{element cross-sectional area}$

Force on element = $\{P(x) - P(x+\Delta x)\} \ A$

$\Rightarrow \rho A \Delta x \frac{\partial E}{\partial x^2} = -A \{P(x+\Delta x) - P(x)\}$ [1 mark]

Dividing by $\Delta x$ and again taking the limit $\Delta x \to 0$,

$\rho \ A \frac{\partial^2 E}{\partial x^2} = -A \frac{\partial P}{\partial x}$

$= -A \frac{\partial}{\partial x} (-E \frac{\partial \varepsilon}{\partial x})$

$\Rightarrow \frac{\partial E}{\partial t^2} = \frac{\rho}{E} \frac{\partial^2 \varepsilon}{\partial x^2}$. [1 mark]

By substitution of $y = f(x-vt)$ or otherwise, we identify the wave speed with

$v = \sqrt{\frac{E}{\rho}}$

so, with $E = 2.32 \times 10^9 \text{ Pa}, \ \rho = 10264 \text{ kg m}^{-3}$,

$v = 1503 \text{ m/s}$. [2 marks]
B4 cont'd. \( E_A = E_B \) so that there are no voids within the medium [1 mark]

\[
E_A \frac{\partial E_A}{\partial x} = E_B \frac{\partial E_B}{\partial x}
\]
i.e. \( P = P_B \) so that the force blows to zero as the volume of any element is reduced to zero - i.e. finite acceleration. [1 mark]

By substitution or otherwise,

\[
\left( \frac{U}{K} \right)^2 = \frac{E}{\rho}
\]

\[
\Rightarrow \quad k = \pm \omega \sqrt{\frac{\rho}{E}}. \quad \text{i.e. } k_{AB} = \pm \omega \sqrt{\frac{\rho_{AB}}{E_{AB}}}
\] [2 marks]

---

Let waves be

1. \( a_i \cos(\omega t - k_A(x-x_0)) \) where \( k_A = \pm \omega \sqrt{\frac{\rho}{E_A}} \)
2. \( a_r \cos(\omega t + k_A(x-x_0)) \)
3. \( a_t \cos(\omega t - k_B(x-x_0)) \)

\[
\Rightarrow \quad E_A(x,t) = a_i \cos(\omega t - k_A(x-x_0)) + a_r \cos(\omega t + k_A(x-x_0)) \\
E_B(x,t) = a_t \cos(\omega t - k_B(x-x_0))
\] [1 mark]
B4 cont'd. Applying the boundary condition at \( x = x_0 \),

\[ a_i \cos \omega t + a_r \cos \omega t = a_t \cos \omega t \]

\[ \Rightarrow a_i + a_r = a_t \quad [1 \text{ mark}] \]

and

\[ E_A a_i k_x \sin \omega t - E_A a_r k_x \sin \omega t = E_A a_t k_x \sin \omega t \]

\[ \Rightarrow E_A b_x (a_i - a_r) = E_A b_x a_t \quad [1 \text{ mark}] \]

Substituting \( a_t = a_i + a_r \) now gives

\[ E_A b_x (a_i - a_r) = E_A b_x (a_i + a_r) \]

\[ \Rightarrow a_i (E_A b_x - E_A b_x) = a_r (E_A b_x + E_A b_x) \]

\[ \Rightarrow \frac{a_r}{a_i} = \frac{E_A b_x - E_A b_x}{E_A b_x + E_A b_x} = \frac{2_A - 2_B}{2_A + 2_B} \quad \text{where } 2 = \sqrt{E} \quad [1 \text{ mark}] \]

Fish neutrally buoyant so \( E_W = \text{Constant} \)

\[ \Rightarrow \left| \frac{a_r}{a_i} \right|^2 = \left( \frac{\sqrt{E_x} - \sqrt{E_W}}{\sqrt{E_x} + \sqrt{E_W}} \right)^2 \quad \text{where } E_x = 2.58 \times 10^6 \text{ Pa} \]

\[ E_W = 2.32 \times 10^6 \text{ Pa} \]

\[ \Rightarrow \text{intensity reflectivity} = \left| \frac{\sqrt{2.58} - \sqrt{2.32}}{\sqrt{2.58} + \sqrt{2.32}} \right|^2 = 0.027^2 = 7 \times 10^{-4} \quad [2 \text{ marks}] \]

Other factors: number of fish in school, thickness of fish (double refraction), homogeneity, size of school, size of fish (diffraction), shape of fish (scattering angles), [2 marks to encourage detailed study]