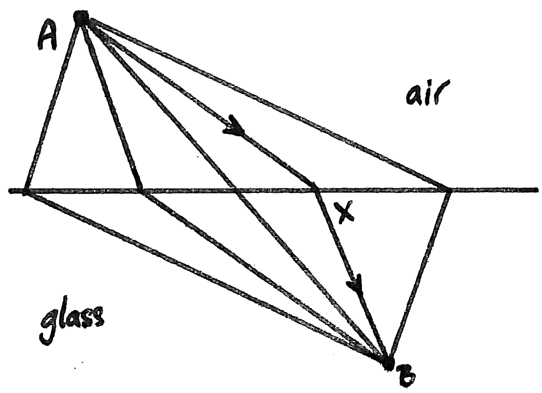
A1. Fermat: in travelling between two points, a wave follows the path which takes the least time. [1 mark]

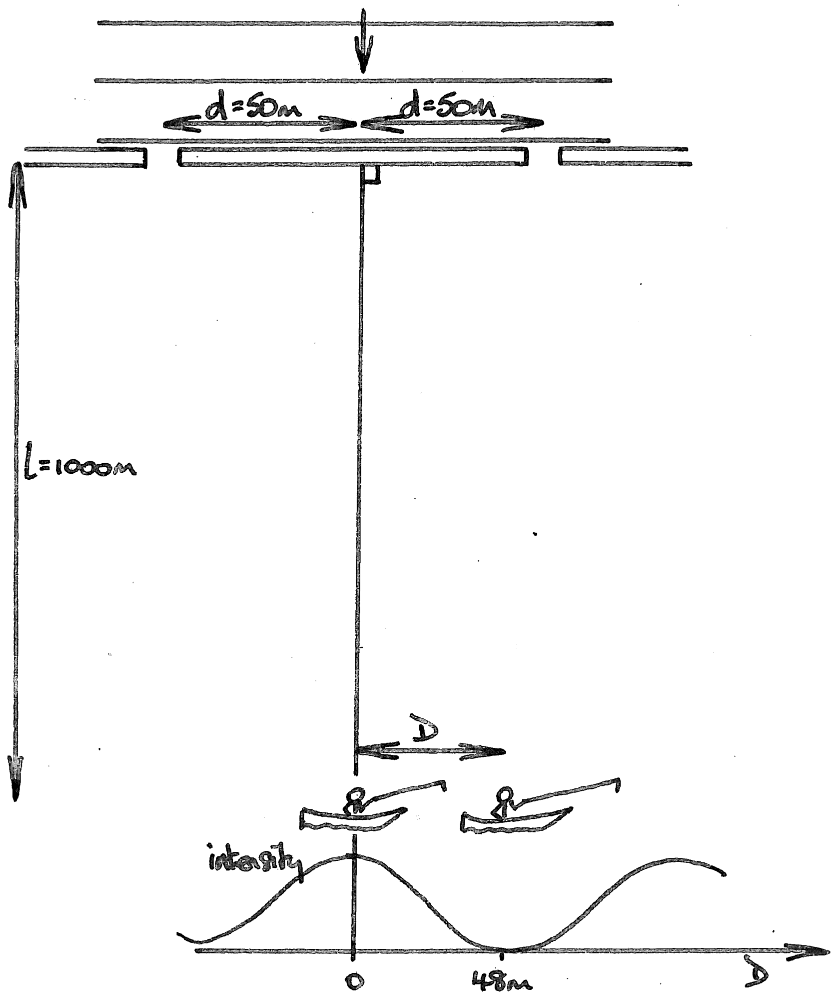


[1 mark]

In comparison with the straight line route, the path via X, although longer, involves spending less time in the slow medium (glass), the saving being greater than the increase in time spent in air to get there. [1 mark]

The path followed will be AXB. [1 mark]

A2. Interference: the addition of wave amplitudes, which may be of opposite sign (destructive interference) or similar sign (constructive interference). As different regions of a wave differ in amplitude and sign, the nature of the interference varies from position to position with the relative timing (phase) of the interfering components. [1 mark]



[1 mark]

Distance of fisherman from gaps in wall $= \sqrt{L^2+(D\pm d)^2}$

$\Rightarrow$ path difference $= \sqrt{L^2+(D+d)^2} - \sqrt{L^2+(D-d)^2}$

which with $L=1000\text{m}$, $D=48\text{m}$, $d=50\text{m}$ gives $1004.79 - 1000.00 = 4.79\text{m}$

For the first minimum, this corresponds to half a wavelength

$\Rightarrow$ wavelength $= 4.79\text{m} \times 2 = \underline{9.58\text{m}}$. [2 marks]

(The small angle approximation is, to this precision, also valid.)

A3. If $\xi(x,t) = \xi_0 \cos(kx - \omega t + \phi)$,

kinetic energy $= \frac{1}{2} \rho \omega^2 \xi_0^2 \sin^2(kx - \omega t + \phi)$

potential energy $= \frac{1}{2} E k^2 \xi_0^2 \sin^2(kx - \omega t + \phi)$

$\Rightarrow$ since $\frac{\omega}{k} \equiv v_p \equiv \sqrt{\frac{E}{\rho}}$, and therefore $\rho\omega^2 = Ek^2$,

the two contributions are equal. [2 marks]

Intensity = wave speed $\times \overline{ke+pe}$

$$= \sqrt{\frac{E}{\rho}} \times \frac{1}{2}\rho\omega^2\xi_0^2 = \sqrt{E\rho}\, \frac{\omega^2 \xi_0^2}{2}$$ [1 mark]

so with $E = 2.3 \times 10^9$ Pa

$\rho = 1025$ kg m^{-3}

$\omega = 50$ kHz $\times 2\pi$,

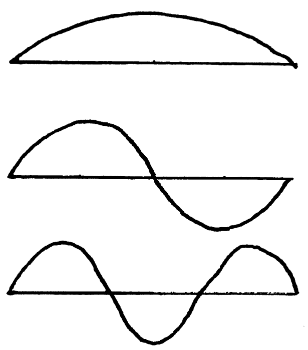
we have 10^{-14} W m^{-2} $= \sqrt{E\rho}\, \frac{\omega^2 \xi_0^2}{2}$

$$\Rightarrow \quad \xi_0 = \sqrt{\frac{2 \times 10^{-14}}{\omega^2 \sqrt{E\rho}}} = \sqrt{\frac{2 \times 10^{-14}}{\pi^2 \times 10^{10} \sqrt{2.3 \times 10^9 \times 1025}}}$$

$= 3.6 \times 10^{-16}$ m. [1 mark]

A4. Boundary conditions are constraints imposed upon the wave at particular positions by the presence of external influences. For example, a guitar string is constrained at the bridge and fret to have a fixed, zero displacement; the air column of a clarinet must have at its open end the unperturbed atmospheric pressure; shallow water waves at a harbour wall may move vertically but not horizontally. [2 marks]

A guitar string is fixed at both ends, and therefore supports sinusoidal standing waves in which the string length is an integer number of half-wavelengths. The corresponding frequencies are therefore integer multiples of the fundamental.

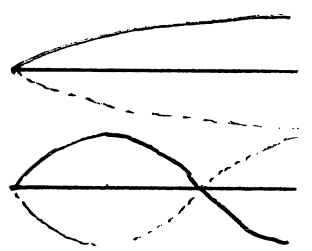


$L = \lambda/2 \quad f_0 = \frac{v}{\lambda} = v/2L$

$L = 2\lambda/2 \quad f = 2\, v/2L = 2f_0$

$L = 3\lambda/2 \quad f = 3\, v/2L = 3f_0$ [1 mark]

The clarinet is closed at the mouthpiece end (displacement = 0) and open to atmospheric pressure at the bell (pressure variation = 0). The pressure depends not upon the displacement but upon its spatial derivative – ie compression rather than simple movement of a given element of the air column. Nodes in the pressure variation hence coincide with anti-nodes of displacement.



$L = \lambda/4 \quad f_0 = \frac{v}{\lambda} = v/4L$

$L = 3\lambda/4 \quad f = 3v/4L = 3f_0$

[1 mark]

Hence the length = odd number of quarter wavelengths, and only odd harmonics are allowed.

A5. Sinusoidal waves: answer should include some of the following:

simplify analysis of differential equations (for linear systems)

provide a complete basis set for the construction of any solution (")

correspond to the emission from sources executing SHM or circular motion

are the eigenmodes of dispersive systems

are what we hear musically, and see as colour

are the basis set corresponding to standing waves (separation of variables)

[2 marks]

$$\exp i(kx-\omega t) \equiv \cos(kx-\omega t) + i\sin(kx-\omega t)$$

$$\cos(kx-\omega t) \equiv \mathrm{Re}\{\exp i(kx-\omega t)\}$$

$$\equiv \frac{1}{2}\{\exp i(kx-\omega t) + \exp -i(kx-\omega t)\}$$

[2 marks]

(or description in terms of superpositions with complex coefficients etc.)

B1. Pressure in element $= \dfrac{-E(\varepsilon_Q - \varepsilon_P)}{\delta x}$ [1 mark]

so, taking the limit $\delta x \to 0$, $P(x) = -E\dfrac{\partial \varepsilon}{\partial x}$ [1 mark]

Mass of element $= \rho\,\delta x\,A$ where A = cross-sectional area of element

Force on element $= \{P(x) - P(x+\delta x)\}A$ [1 mark]

$\Rightarrow \rho A\,\delta x\,\dfrac{\partial^2 \varepsilon}{\partial t^2} = -A\{P(x+\delta x) - P(x)\}$ [1 mark]

Dividing by δx and again taking the limit $\delta x \to 0$,

$$\rho A \frac{\partial^2 \varepsilon}{\partial t^2} = -A\frac{\partial P}{\partial x}$$

$$= -A\frac{\partial}{\partial x}\left(-E\frac{\partial \varepsilon}{\partial x}\right)$$

$$\Rightarrow \underline{\underline{\frac{\partial^2 \varepsilon}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 \varepsilon}{\partial x^2}.}}$$ [1 mark]

By substitution of $y = f(x - vt)$ or otherwise, we identify the wave speed with

$$v = \sqrt{E/\rho}$$

so, with $E = 193 \times 10^9$ Pa, $\rho = 8030\,\mathrm{kg\,m^{-3}}$,

$$\underline{\underline{v = 4{,}900\ \mathrm{m\,s^{-1}}.}}$$ [2 marks]

B1 cont'd. $\varepsilon_A = \varepsilon_B$ so that there are no voids within the medium [1 mark]

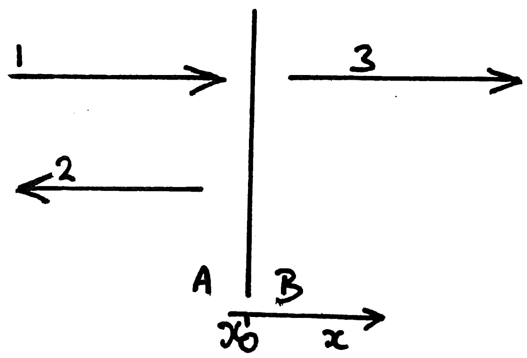
$E_A \dfrac{\partial \varepsilon_A}{\partial x} = E_B \dfrac{\partial \varepsilon_B}{\partial x}$ i.e. $P_A = P_B$ so that the force falls to zero as the volume of any element is reduced to zero – i.e. finite acceleration.

[1 mark]

By substitution or otherwise,

$$\left(\frac{\omega}{k}\right)^2 = \frac{E}{\rho}$$

$$\Rightarrow \quad \underline{\underline{k = \pm \omega \sqrt{\frac{\rho}{E}}}} . \qquad \text{i.e. } \underline{\underline{k_{A,B} = \pm \omega \sqrt{\frac{\rho_{A,B}}{E_{A,B}}}}}$$ [2 marks]



let waves be

1 $\quad a_i \cos(\omega t - k_A(x - x_0))$ where $k_A = +\omega\sqrt{\frac{\rho_A}{E_A}}$

2 $\quad a_r \cos(\omega t + k_A(x - x_0))$

3 $\quad a_t \cos(\omega t - k_B(x - x_0))$ [1 mark]

$$\Rightarrow \varepsilon_A(x,t) = a_i \cos(\omega t - k_A(x - x_0)) + a_r \cos(\omega t - k_A(x - x_0))$$

$$\varepsilon_B(x,t) = a_t \cos(\omega t - k_B(x - x_0))$$ [1 mark]

B1 cont'd. Applying the boundary conditions at $x = x_0$,

$$a_i \cos\omega t + a_r \cos\omega t = a_t \cos\omega t$$

$$\Rightarrow \quad a_i + a_r = a_t \qquad \text{[1 mark]}$$

and $E_A a_i k_A \sin\omega t - E_A a_r k_A \sin\omega t = E_A a_t k_B \sin\omega t$

$$\Rightarrow \quad E_A k_A (a_i - a_r) = E_B k_B a_t \qquad \text{[1 mark]}$$

Substituting $a_t = a_i + a_r$ now gives

$$E_A k_A (a_i - a_r) = E_B k_B (a_i + a_r)$$

$$\Rightarrow \quad a_i (E_A k_A - E_B k_B) = a_r (E_A k_A + E_B k_B)$$

$$\Rightarrow \quad \frac{a_r}{a_i} = \frac{E_A k_A - E_B k_B}{E_A k_A + E_B k_B} = \underline{\underline{\frac{Z_A - Z_B}{Z_A + Z_B}}} \quad \text{where } Z \equiv \sqrt{E\rho}. \qquad \text{[1 mark]}$$

$\Rightarrow$ for the fluid-tissue interface,

$$\left|\frac{a_r}{a_i}\right|^2 = \left|\frac{\sqrt{E_F \rho_F} - \sqrt{E_T \rho_T}}{\sqrt{E_F \rho_F} + \sqrt{E_T \rho_T}}\right|^2$$

where $E_F = 1.5 \times 10^9$ Pa
$E_T = 2.7 \times 10^9$ Pa
$\rho_F = 1000$ kgm^{-3}
$\rho_T = 1070$ kgm^{-3}

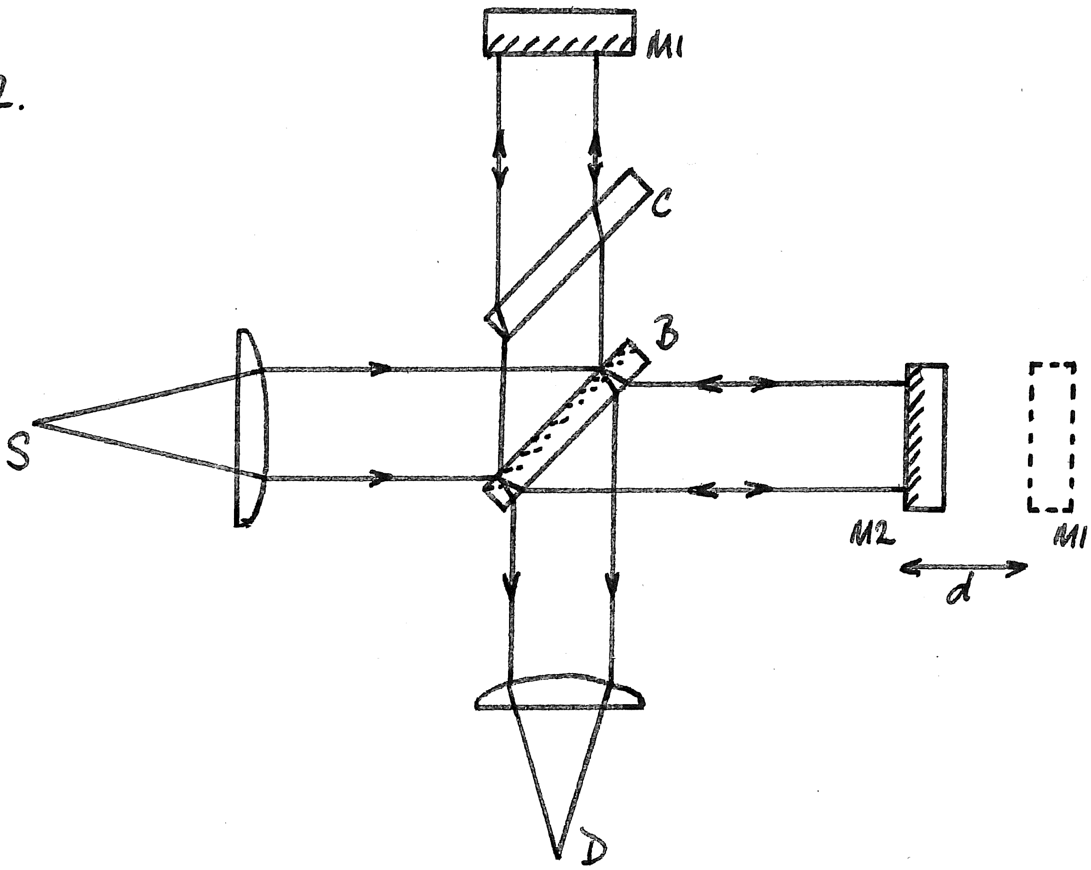
$$\Rightarrow \text{intensity reflectivity} = \left|\frac{\sqrt{1.5} - \sqrt{2.7 \times 1.07}}{\sqrt{1.5} + \sqrt{2.7 \times 1.07}}\right|^2$$

$$= 0.162^2 = \underline{\underline{0.026.}} \qquad \text{[2 marks]}$$

Other factors: homogeneity
size of feature (diffraction)
orientation (specular reflection)
shape (scattering angles)
thickness (double reflection)
absorption (scattering in path.

[2 marks for something sensible]

B2.

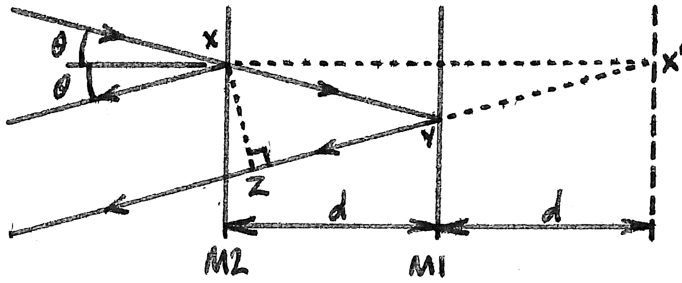


[2 marks]

The Michelson interferometer works by interfering two beams of light, obtained from the same source by division of amplitude, which reach the detector by paths of different length.
Collimated light from the source S strikes the partially-reflecting beam-splitter B. Part of the light is reflected, and passes to mirror M1, reflecting it back along its path; the other part is transmitted, passes to M2, and again is reflected back to the beam-splitter. Here, the two beams are recombined, being respectively transmitted and reflected to reach the detector D. Depending upon the displacement d from the position of equal path length M1′, the movable mirror M2 introduces a path difference between the two routes, which results in constructive or destructive interference according to d.

[3 marks]

B2 contd.



path difference $= XYZ$ [1 mark]

$= X'YZ$ where X' is reflection of X in mirror $M1$

$= 2d\cos\theta$

$= s\cos\theta$ if s = path difference at normal incidence [1 mark]

$\Rightarrow$ if incident light varies as $E(x) = E_0 \exp i(kx-\omega t)$, then the two components may be written $E(x_0)$, $E(x_0 + s\cos\theta)$

ie. $\quad E_0 \exp i(kx_0-\omega t)$, $\quad E_0 \exp i(kx_0 + ks\cos\theta - \omega t)$

or, in terms of a common factor,

$$\underline{\underline{\begin{array}{l} E_0 \exp i(k(x_0-\tfrac{s}{2}\cos\theta-\omega t) \;\; \exp ik\tfrac{s}{2}\cos\theta \\ E_0 \exp i(k(x_0-\tfrac{s}{2}\cos\theta-\omega t) \;\; \exp -ik\tfrac{s}{2}\cos\theta \end{array}}}$$ [2 marks]

Thus the total electric field of the superposition will have the form

$$E_0 \exp i(k(x_0-\tfrac{s}{2}\cos\theta)-\omega t))\left(e^{ik\frac{s}{2}\cos\theta} + e^{-ik\frac{s}{2}\cos\theta}\right)$$

$$= 2E_0 \exp i(k(x_0-\tfrac{s}{2}\cos\theta)-\omega t) \;\; \cos\left(\tfrac{ks}{2}\cos\theta\right)$$ [1 mark]

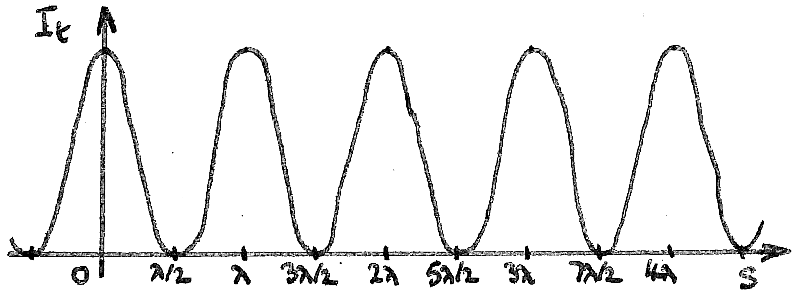
and the transmitted intensity, proportional to the squared magnitude of the field, will be proportional to $\left[\cos\left(\frac{ks}{2}\cos\theta\right)\right]^2$

ie. $\underline{\underline{I_t \propto \cos^2\left(\frac{ks}{2}\cos\theta\right).}}$ [1 mark]

B2 contd. (a) single frequency He-Ne, $\lambda = 488\,\mathrm{nm}$

$$\Rightarrow I_t \propto \cos^2 \frac{ks}{2} \quad \text{with } k = \frac{2\pi}{\lambda}.$$

* continuous sinusoidal variation
* period $= \lambda = 488\,\mathrm{nm}$



[3 marks]

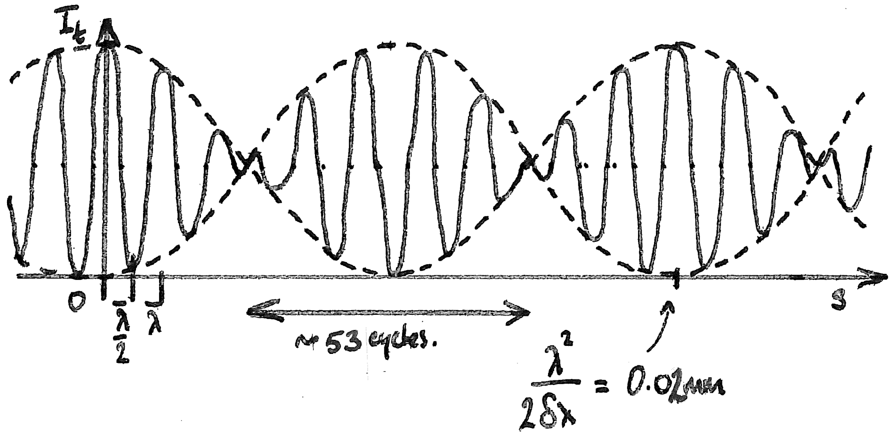
(b) With two wavelengths $\lambda_1 = 780.2\,\mathrm{nm}$ and $\lambda_2 = 795.0\,\mathrm{nm}$

$$I_t \propto \cos^2 \frac{k_1 s}{2} + \cos^2 \frac{k_2 s}{2} \quad \text{where } k_{1,2} = \frac{2\pi}{\lambda_{1,2}}$$

$$\cos^2\theta \equiv \tfrac{1}{2}(1+\cos 2\theta)$$

$$\Rightarrow \quad I_t \propto 1 + (\cos k_1 s + \cos k_2 s)/2$$

$$= 1 + \cos \frac{k_1+k_2}{2} s \, \cos \frac{k_1-k_2}{2} s$$



[3 marks]

*Here drawn schematically: note real scales of two periods.

B2 contd. If r, t are not equal, the two paths nonetheless contribute equally to the output beam, since each path involves one reflection and one transit through the beamsplitter. The visibility and extinction of the fringe pattern observed is therefore unchanged. [2 marks]

Light not transmitted by the instrument is reflected back to the source. [1 mark]

B3. Dispersion is the variation of wave speed with frequency. For waves composed of several frequency components, it results in a change in the shape of the wave as it propagates. [2 marks]

The phase velocity is the velocity of propagation of a point of constant phase (i.e. a wavefront) in a single frequency (sinusoidal) wave. The group velocity is the velocity of propagation of a wavepacket, i.e. the envelope or intensity profile of a superposition of sinusoidal components. [2 marks]

phase velocity $v_p = \omega/k$

group velocity $v_g = \frac{d\omega}{dk}$

[2 marks]

To combine the equations, we first differentiate them w.r.t. t and x respectively:

$$\frac{\partial^2 h}{\partial t^2} = -h_0 \frac{\partial^2 v}{\partial t \partial x}$$

$$\frac{\partial^2 v}{\partial t \partial x} = -g \frac{\partial^2 h}{\partial x^2}$$

[2 marks]

and now substitute to eliminate $\partial^2 v/\partial t \partial x$:

$$\underline{\underline{\frac{\partial^2 h}{\partial t^2} = g h_0 \frac{\partial^2 h}{\partial x^2}.}}$$

[1 mark]

The form of this equation (or direct substitution of $h = H\cos(kx - ct + \phi)$) gives the phase velocity

$$\underline{\underline{v_p = \sqrt{g h_0}}}$$

hence $\omega = \sqrt{g h_0}\, k$, so

$$\underline{\underline{v_g = \sqrt{g h_0}}} \text{ also.}$$

[2 marks]

B3 cont'd. For shallow water waves, $\omega = \sqrt{gk}$, so

$$v_p = \omega/k = \frac{\sqrt{gk}}{k} = \underline{\underline{\sqrt{g/k}}} \text{ as required,}$$ [1 mark]

and $$v_g = \frac{d\omega}{dk} = \frac{d}{dk}\sqrt{gk} = \underline{\underline{\frac{1}{2}\sqrt{g/k}.}}$$ [1 mark]

Thus for shallow water waves, the phase and group velocities are different ($v_g = v_p/2$) and vary with frequency. [2 marks]

Reinforced waves satisfy

$$v_p(\omega) = v_{boat}\cos\theta$$

so, inserting $v_p(\omega)$,

$$\sqrt{g/k} = v_{boat}\cos\theta$$

$$\Rightarrow \sqrt{\frac{g\lambda}{2\pi}} = v_{boat}\cos\theta$$

$$\text{so} \quad \underline{\underline{\lambda = \frac{2\pi}{g}(v_{boat}\cos\theta)^2.}}$$ [2 marks]

For waves travelling in the same direction as the boat, $\theta = 0$, so

$$\underline{\underline{\lambda = \frac{2\pi v_{boat}^2}{g}.}}$$ [1 mark]

At the hull speed, $\lambda = L$, so

$$v_{hull} = \sqrt{\frac{Lg}{2\pi}} = \sqrt{\frac{17.3 \; 9.81}{2\pi}} \text{ m s}^{-1} = \underline{\underline{5.20 \text{ m s}^{-1}.}}$$ [2 marks]

[The Oxford-Cambridge course is 4 miles 374 yards ≡ 6784 m; the 2008 winning team took 20 mins 53 s (considered very slow), giving an average speed of 5.4 m s^{-1}.]

B4. The Fourier transform allows a function of time or position to be instead represented as a function of frequency or spatial frequency – ie. by the spectrum of sinusoidal or complex exponential components into which it may be resolved. [2 marks]

The component with a given frequency is obtained by multiplying the function by a sinewave (or complex exponential wave) with the same (or negative) frequency, and integrating over the range of the function,

eg.
$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$f(\omega) = \int_{-\infty}^{\infty} f(t) \cos \omega t \, dt \quad \text{etc.}$$ [2 marks]

(The overall factor of $\frac{1}{\sqrt{2\pi}}$ is an arbitrary choice, determined by whether the aim is to symmetrize the Fourier transform and its inverse or to normalize the spectral intensity.)

The symmetry of $a(t)$ allows it to be expressed as a superposition of cosine waves $\cos \omega t$ of amplitudes $b(\omega)$, where

$$b(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(t) \cos \omega t \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-T/2}^{T/2} \cos \omega_0 t \cos \omega t \, dt$$ [1 mark]

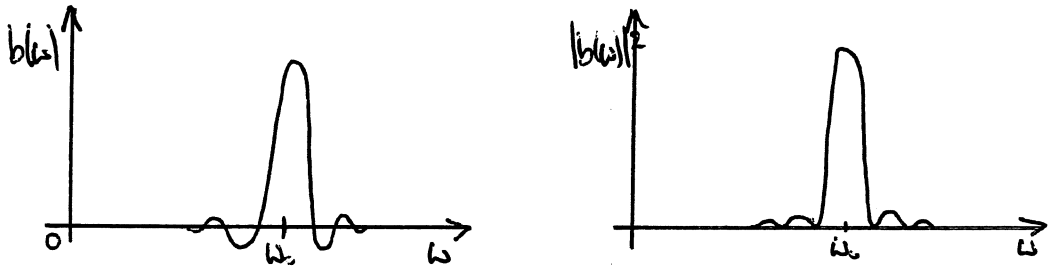
$$= \frac{1}{\sqrt{2\pi}} \int_{-T/2}^{T/2} \frac{1}{2}\left(\cos(\omega_0 + \omega)t + \cos(\omega_0 - \omega)t\right) dt$$ [1 mark]

$$= \frac{1}{2\sqrt{2\pi}} \left[\frac{\sin(\omega_0 + \omega)t}{\omega_0 + \omega} + \frac{\sin(\omega_0 - \omega)t}{\omega_0 - \omega} \right]_{-T/2}^{T/2}$$ [1 mark]

B4 cont'd $\Rightarrow b(\omega) = \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(\omega_0+\omega)T/2}{\omega_0+\omega} + \frac{\sin(\omega_0-\omega)T/2}{\omega_0-\omega}\right]$

Since $\omega_0 T \gg 1$, for frequencies around $\omega = \omega_0$ the second term dominates

$$\Rightarrow b(\omega) \approx \frac{T/2}{\sqrt{2\pi}} \frac{\sin(\omega_0-\omega)T/2}{(\omega_0-\omega)T/2}.$$ [1 mark]



[4 marks]

At half-maximum intensity, $|b(\omega)|^2 = \frac{1}{2}|b(\omega_0)|^2$ [1 mark]

$$\Rightarrow \left|\frac{\sin(\omega_0-\omega)T/2}{(\omega_0-\omega)T/2}\right|^2 = 1/2$$

i.e. $\frac{\sin^2 a}{a^2} = \frac{1}{2}$ where $a = \pm\frac{(\omega_0-\omega)T}{2}$

$$\Rightarrow \frac{(\omega_0-\omega)T}{2} = \pm 1.392$$

$$\Rightarrow |\omega_0-\omega| = \frac{2\ 1.392}{T}$$ [2 marks]

$$\Rightarrow \text{FWHM} = 2\,\frac{2\ 1.392}{T} = \frac{5.568}{T}\ \text{rads.sec}^{-1} \equiv \frac{0.886}{T}\ \text{Hz}.$$ [1 mark]

B4 cont'd. The difference in delays between pulses reflected from aircraft differing in distance by 15 m = Δh will be $2\Delta h/c$, where c is the speed of propagation. It seems reasonable that the pulse length T should be less than this,

ie. $T \lesssim \frac{2\Delta h}{c}$ $\quad (= 10^{-7}\,\mathrm{s})$ [2 marks]

so that the spectrum of the pulse will satisfy

$$\mathrm{FWHM} = \frac{0.886}{T} \gtrsim \frac{0.886\,c}{2\Delta h}$$

If $c = 3\times10^{8}\,\mathrm{m.s^{-1}}$, $\Delta h = 15\,\mathrm{m}$, we thus obtain

$$\underline{\underline{\mathrm{FWHM} \gtrsim 8.9\ \mathrm{MHz} \approx 10\,\mathrm{MHz}.}}$$ [2 marks]

For negligible pulse distortion, the microwave transducers must have a bandwidth of at least around this value.

(There are several alternative criteria - eg. requiring a certain steepness to the pulse edge - which will all give answers of this order of magnitude.)