SEMESTER 1 EXAMINATION 2015-2016

WAVE PHYSICS

Duration: 120 MINS (2 hours)

This paper contains 9 questions.

Answers to Section A and Section B must be in separate answer books

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language Word to Word® translation dictionary (paper version) is permitted provided it contains no notes, additions or annotations.

Section A

A1. Mountain lee waves are transverse (vertical) motions of the atmosphere when a strong wind blows over a mountain or hill; they occur downwind of the mountain, extend to all altitudes, and are beloved of glider pilots who use them to soar to heights of many kilometres. The vertical motion of the atmospheric airmass can be taken to be roughly sinusoidal,

$$\psi(x,t) = \psi_0 \cos(kx) = \psi_0 \cos(kx' + \omega t),$$

where $\psi(x,t)$ is the vertical displacement of the airmass from its initial altitude, x is its horizontal distance downwind of the mountain, $x' = x - v_{\rm wind}t$ is its position within the moving airmass as it travels with speed $v_{\rm wind}$ relative to the ground, and $\omega = v_{\rm wind}k$.

Show that the vertical velocity of the air at position x and time t is given by

$$v_{\rm vert} = -\psi_0\, v_{\rm wind}\, k \sin(kx). \qquad [\,2\,]$$

The (element of the) airmass is defined by the coordinate x', so its vertical velocity will be *[2]*

$$v_{\rm vert} = \left.\frac{\partial\psi}{\partial t}\right|_{x'} = -\psi_0\,\omega\sin(kx' + \omega t) = -\psi_0\, v_{\rm wind}\, k \sin[k(x - v_{\rm wind}t) + \omega t] = -\psi_0\, v_{\rm wind}\, k\sin(kx).$$

A glider requires the air to rise with a minimum speed of 1 m s^{-1} for it to remain aloft. If $v_{\rm wind}$ = 10 m s^{-1} and $2\pi/k = 5{,}000\,\rm m$, find the minimum amplitude ψ_0 of the wave motion. [2]

Assuming that the glider can find the position of greatest vertical velocity, we require

$$\left.\frac{\partial\psi}{\partial t}\right|_{x'}^{max} \equiv \psi_0\, v_{\rm wind}\, k \geq 1\,\rm m\,s^{-1}$$

hence *[2]*

$$\psi_0 \geq \frac{1\,\rm m\,s^{-1}}{v_{\rm wind}\,k} = \frac{1\,\rm m\,s^{-1}\;5{,}000\,m}{10\,\rm m\,s^{-1}\;2\pi} \approx 80\,\rm m.$$

A2. Describe the Doppler effect, and discuss two methods by which it may be theoretically derived. [4]

The Doppler effect is the shift in frequency recorded when there is a relative motion between the observer and the wave source. *[2]*

The classical derivation of the Doppler effect considers the progression of propagating wavefronts relative to the motion of the source and observer, and establishes the times and positions when the wavefronts and source or observer meet [1 and/or...]. The Doppler shift also emerges automatically from the transformations of special relativity [1 and/or...], and from quantum-mechanical treatments, in which waves have quantized energies and momenta, from the principles of conservation of energy and momentum [1]. *[2]*

A3. Explain why a laser pulse must comprise a range of optical frequencies, and describe how the frequency range is related to the length of the pulse. [4]

A single frequency waveform has a steady amplitude, so for the intensity to vary there must be different frequency components whose relative phase, and hence the constructive or destructive nature of their interference, will change with time. *[2]*

For a faster variation in intensity, the phase difference must evolve more quickly, requiring a greater difference in frequency. The minimum bandwidth of the spectrum is therefore inversely proportional to the pulse length. *[2]*

A4. The antenna for a stereo radio receiver comprises a metal rod (cut at the middle for the connection to the radio), which acts as a resonator for radio waves of frequency 100 MHz. By considering the boundary conditions at the ends of the rod, and assuming the speed of electromagnetic waves along the rod to equal the speed of light in vacuum, c, calculate the shortest total length of the rod. [3]

The antenna is symmetrical, so the boundary conditions will be the same at each end [1]. The length of the rod must therefore be an integer number of half-wavelengths [1]. The shortest length is therefore $0.5c/100\,\mathrm{MHz} = 1.5\,\mathrm{m}$ *[1].* *[3]*

At which other frequencies will the antenna also act as a resonator? [1]

The antenna will also be a resonator when the length is a higher integer multiple N *of the half-*

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wavelength, i.e. $0.5Nc/f = 1.5\,\mathrm{m}$ *where* f *is the frequency, giving* $f = N \times 100\,\mathrm{MHz}$. *[1]*

A5. Explain what are meant by the terms *travelling* and *standing* waves. [2]

Travelling waves are those which maintain a constant form that is simply translated through space as time evolves [1]. Standing waves maintain a spatially fixed form, that is multiplied by an evolving function of time [1]. *[2]*

Show, with an example, how travelling waves may be superposed to form a standing wave, and vice-versa. [2]

It helps to take a sinusoidal or complex exponential wave:

$$\begin{aligned} A\cos kx \cos \omega t &\equiv \frac{A}{2}\left[\cos(kx - \omega t) + \cos(kx + \omega t)\right] \\ B\cos(kx - \omega t) &\equiv B\left[\cos kx \cos \omega t + \sin kx \sin \omega t\right]. \end{aligned}$$

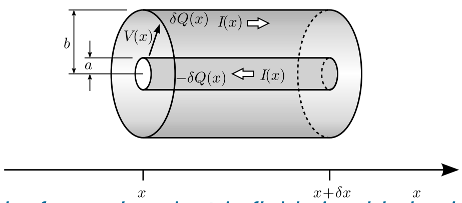
[1 mark for each.] *[2]*

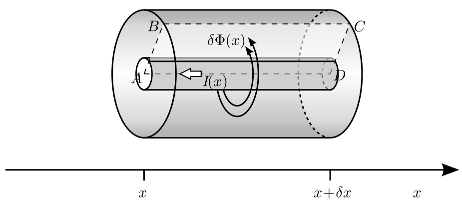
Section B

B1. A coaxial cable comprises an inner conductor of radius a that lies concentrically within a cylindrical outer conductor of radius b. The space between the two conductors is filled with a non-magnetic dielectric of relative permittivity ε. The capacitance and inductance per unit length, C and L, are given by

$$C = \frac{2\pi\varepsilon_0\varepsilon}{\ln(b/a)},$$
$$L = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right).$$

(a) Sketch the geometry of the coaxial cable, indicating the voltages $V(x,t)$ and currents $I(x,t)$, charges $Q(x,t)$, and electric and magnetic fields $E(x,t)$ and $B(x,t)$, for an element of length δx at position x and time t. [4]





[2 marks for each - electric field should also be shown] *[4]*

(b) Derive the relationship

$$C\frac{\partial V}{\partial t} = -\frac{\partial I}{\partial x}.$$ [3]

From conservation of charge, with $\pm\delta Q$ the charge on each conductor in the element of length δx,

$$\frac{\partial}{\partial t}\delta Q(x) = I(x) - I(x+\delta x).$$ *[1]*

From the definition of capacitance per unit length,

$$\delta Q(x) = (C\,\delta x)\,V(x).$$ *[1]*

Combining these equations and taking the limit as $\delta x \to 0$,

$$C\frac{\partial V}{\partial t} = \lim_{\delta x\to 0}\frac{I(x)-I(x+\delta x)}{\delta x} = -\frac{\partial I}{\partial x}.$$ *[1]*

(c) Derive the relationship

$$L\frac{\partial I}{\partial t} = -\frac{\partial V}{\partial x}.$$ [3]

From Faraday's law, with $\delta\Phi$ the magnetic flux between the conductors in the element of length δx,

$$\frac{\partial}{\partial t}\delta\Phi(x) = V(x) - V(x+\delta x).$$ *[1]*

From the definition of inductance per unit length,

$$\delta\Phi(x) = (L\,\delta x)\,I(x).$$ *[1]*

Combining these equations and taking the limit as $\delta x \to 0$,

$$L\frac{\partial I}{\partial t} = \lim_{\delta x\to 0}\frac{V(x)-V(x+\delta x)}{\delta x} = -\frac{\partial V}{\partial x}.$$ *[1]*

(d) Hence derive the wave equation

$$\frac{\partial^2 V}{\partial t^2} = \frac{1}{LC}\frac{\partial^2 V}{\partial x^2},$$ [3]

Differentiating our first two expressions with respect to time and position respectively, we obtain

$$\begin{aligned} C\frac{\partial^2 V}{\partial t^2} &= -\frac{\partial^2 I}{\partial x\partial t} \\ L\frac{\partial^2 I}{\partial t\partial x} &= -\frac{\partial^2 V}{\partial x^2} \end{aligned}$$

which may be combined to eliminate $\partial^2 I/\partial x\partial t$ to give

$$\frac{\partial^2 V}{\partial t^2} = -\frac{1}{C}\frac{\partial^2 I}{\partial x\partial t} = \frac{1}{LC}\frac{\partial^2 V}{\partial x^2}.$$

[1 mark for each expression] *[3]*

and give an expression for the phase velocity of electromagnetic waves along the cable. [1]

By inspection or substitution of a trial solution, we identify the phase velocity from the wave equation as $v = 1/\sqrt{LC}$. *[1]*

(e) Given that the diameters of the inner and outer conductors are 1 mm and 5 mm respectively, and the space between them is filled with solid PTFE with a relative permittivity of 2.05, find the capacitance and inductance per unit length. [4]

$$C = \frac{2\pi\varepsilon_0\varepsilon}{\ln(b/a)} = \frac{2\pi\, 8.85\times10^{-12}\, 2.05}{\ln(5/1)} = 70\,\mathrm{pF\,m^{-1}}$$
$$L = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right) = \frac{4\pi\times10^{-7}}{2\pi}\ln\left(\frac{5}{1}\right) = 320\,\mathrm{nH\,m^{-1}}.$$

[2 marks for each] *[4]*

(f) Hence determine the time taken for a signal to propagate along a 1 m length of the cable. [2]

The phase velocity will be $v = 1/\sqrt{LC} = 1/\sqrt{70\,10^{-12}\times 320\,10^{-9}} = 2.11\times10^{8}\,\mathrm{m\,s^{-1}}$. *The time to travel 1 m is hence* $1\,\mathrm{m}/2.11\times10^{8}\,\mathrm{m\,s^{-1}} = 4.8\,\mathrm{ns}$. *[2]*

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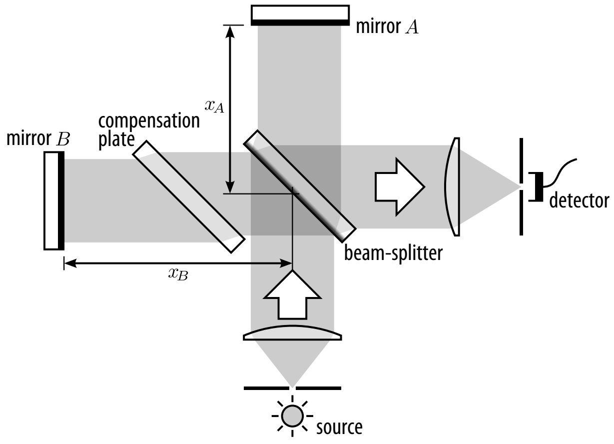
B2. (a) Sketch and explain the operation of the *Michelson interferometer*. [5]



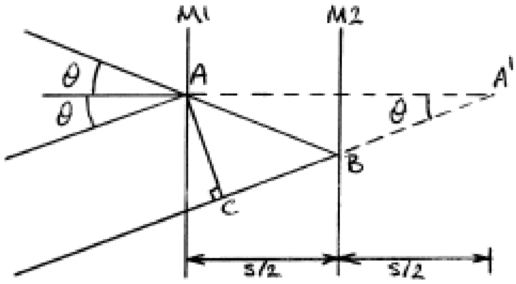
[2]

The Michelson interferometer works by interfering two beams of light, obtained from the same source by division of amplitude, which reach the detector by paths of different length. *[1]*

Collimated light from the source strikes the partially-reflecting beam-splitter. Part of the light is transmitted, and passes to mirror A, which reflects it back along its path; the other part is reflected by the beam-splitter, travels to mirror B, and again is reflected back. The two beams are recombined at the beam-splitter, being respectively reflected and transmitted to reach the detector. Depending upon the displacement of the movable mirror from the position of equal path length $x_A = x_B$, there is a path difference $2(x_B - x_A)$ between the two routes, which results in constructive or destructive interference according to the path difference. *[2]*

(The existence and rôle of the compensation plate need not be described.)

(b) The amplitude of the light transmitted by a Michelson interferometer may be determined by summing the amplitudes resulting from the two routes through the interferometer. Given that the partially-reflecting beamsplitter divides incident light equally between the two paths, the difference in path length between the two routes at normal incidence is s, and the rays of wavelength λ make an angle θ to the mirror normals, sketch the geometry and write complex exponential expressions for the relative amplitudes of these two contributions. [4]



[2]

The figure above shows a comparison of two paths through the instrument, at the same angle ϑ to the mirror normals, via the two mirror positions M_1 and M_2; in reality, the two paths are followed in separate arms of the instrument. The path difference will be the distance ABC = A'BC = $s\cos\vartheta$, where s is the path difference at normal incidence.

The two contributions to the final field amplitude may hence be written a_0 and $a_0\exp(-iks\cos\vartheta)$, where the wavenumber $k \equiv 2\pi/\lambda$; equivalently, writing $a \equiv a_0\exp(-iks\cos\vartheta/2)$, the contributions may be written more symmetrically as

$$a\exp(\pm iks\cos\vartheta/2)\,.$$ *[2]*

(c) Hence show that the overall intensity transmitted by the interferometer is given by

$$I_\mathrm{t} \propto \cos^2\left(\frac{k\,s}{2}\cos\theta\right),$$

where I_t is the transmitted intensity and $k = 2\pi/\lambda$. [2]

The combined amplitude will be

$$a\left[\exp(iks\cos\vartheta/2) + \exp(-iks\cos\vartheta/2)\right] = 2a\cos\left(\frac{ks\cos\vartheta}{2}\right),$$ *[1]*

where for non-absorbing mirrors $|2a| = 1$.

The transmitted intensity – proportional to the square of the amplitude – will therefore be

$$I_\mathrm{t} \propto \cos^2\left(\frac{ks\cos\vartheta}{2}\right).$$ *[1]*

(d) A Michelson interferometer is used to investigate the spectra of a number of light sources, by recording the on-axis transmitted intensity I_t as a

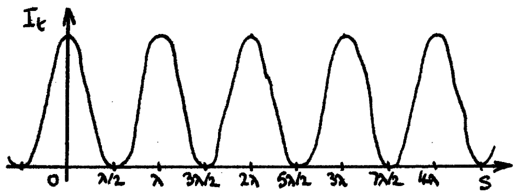
TURN OVER

function of the path difference s. Sketch, and label with as much detail as you can, the *interferograms* (recordings of I_t vs s) that would be obtained when the light source is

(i) a single-frequency telecommunications laser with an infrared wavelength of 1561.4 nm. [3]

(ii) a low-pressure potassium lamp, which principally emits two wavelengths of 766.5 nm and 769.9 nm, taking them to be equal in intensity. [3]

(i) For a single frequency, $I_t \propto \cos^2 ks/2$, with $k = 2\pi/\lambda$, so the interferogram will be regular fringes of period $\lambda = 1561.4$ nm.



[3]

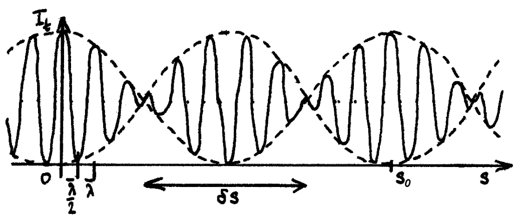
(ii) With two wavelengths $\lambda_{1,2}$, the interferograms of the individual components will be superposed, i.e.

$$I_t \propto \cos^2 \frac{k_1 s}{2} + \cos^2 \frac{k_2 s}{2}$$

where $k_{1,2} = 2\pi/\lambda_{1,2}$. Since $\cos^2 \vartheta \equiv (1 + \cos 2\vartheta)/2$, this gives

$$I_t \propto 1 + (\cos k_1 s + \cos k_2 s)/2 = 1 + \cos \frac{(k_1 + k_2)s}{2} \cos \frac{(k_1 - k_2)s}{2},$$

as shown schematically below.



[3]

Here, $(k_1 - k_2)\delta s/2 = \pi$, so $\delta s = (1/\lambda_1 - 1/\lambda_2)^{-1}$, and $s_0 = 2\delta s$.
The interval δs spans $[(k_1 + k_2)\delta s/2]/(2\pi) \approx \delta s/\lambda_{1,2}$ fringes.
For the potassium lines, $\delta s = 174\,\mu$m, $s_0 = 347\,\mu$m and the interval shown spans 226 fringes.

(e) How is the transmission of the interferometer modified if the amplitude transmission t and reflectivity r of the partially-reflecting beamsplitter are not equal? What happens to the fraction of light that is not transmitted by the instrument? [3]

Even if r and t are not equal, the two paths nonetheless contribute equally to the output beam, since each path involves one reflection and one transit through the beam-splitter. The visibility and extinction of the fringe pattern observed is therefore unchanged (although its intensity will be lower). *[2]*

(Note that this is not true of the reflected pattern, which is reduced in visibility.)

Light not transmitted by the instrument is reflected back to the source. *[1]*

TURN OVER

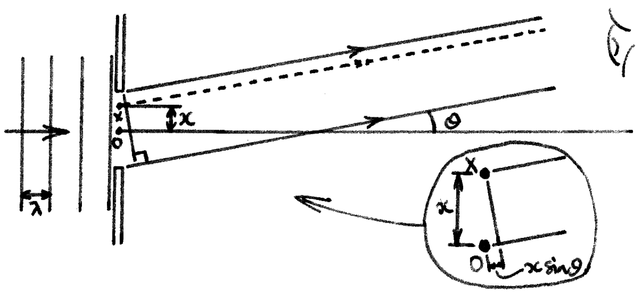
B3. (a) Explain what is meant by *Fraunhofer diffraction*, and give an example of its rôle or occurrence. [4]

Fraunhofer diffraction is the effect upon a propagating wave of an opaque or refractive mask or obstruction, when viewed in the image plane of the wave source – for example, when plane waves are viewed from infinity. *[2]*

Examples include the use of diffraction gratings for spectroscopy, the limit upon the resolution of an optical instrument due to the finite apertures of its optical elements, the vivid colours of butterfly wings, x-ray diffraction analysis, and the patterns apparent when a small, distant source is observed through a finely woven fabric. *[2]*

(b) Show from first principles that $p(\vartheta)$, the dependence upon angle of the relative amplitude of light of wavelength λ, when diffracted by a slit of width a, is proportional to the sinc function

$$p(\vartheta) \propto \frac{\sin\{\frac{\pi a}{\lambda}\sin\vartheta\}}{\frac{\pi a}{\lambda}\sin\vartheta}.$$ [4]



We consider light, arriving at the slit as plane waves, passing via point X a distance x from the slit centre, and proceeding at an angle ϑ to the distant observer. The path length differs from that through the slit centre O by $x\sin\vartheta$, corresponding to a phase difference

$$\delta\varphi = \frac{2\pi}{\lambda}x\sin\vartheta.$$ *[1]*

The contribution to the diffracted amplitude is thus

$$a_0 \exp i\delta\varphi = a_0 \exp\left(i\frac{2\pi\sin\vartheta}{\lambda}x\right).$$ *[1]*

The total diffracted amplitude is found by summing the contributions via all points in the slit

$$p(\vartheta) = \int_{-a/2}^{a/2} a_0 \exp\left(i\frac{2\pi\sin\vartheta}{\lambda}x\right) \mathrm{d}x$$

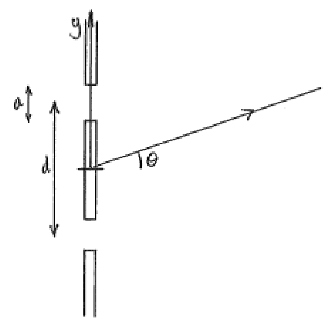
$$
\begin{aligned}
&= \frac{a_0}{i\frac{2\pi\sin\vartheta}{\lambda}}\left[\exp\left(i\frac{2\pi\sin\vartheta}{\lambda}x\right)\right]_{-a/2}^{a/2} \\
&= \frac{2a_0}{i\frac{2\pi\sin\vartheta}{\lambda}}\frac{\exp\left(i\frac{2\pi\sin\vartheta}{\lambda}\frac{a}{2}\right)-\exp\left(-i\frac{2\pi\sin\vartheta}{\lambda}\frac{a}{2}\right)}{2i} \\
&= a_0 a\frac{\sin\frac{\pi a\sin\vartheta}{\lambda}}{\frac{\pi a\sin\vartheta}{\lambda}}
\end{aligned}
$$

as required. *[2]*

(c) Two long, transmitting slits, each of width a, are separated by a non-transmitting region of width $(d-a)$. Show that the Fraunhofer diffraction pattern - *i.e.* the intensity distribution of the light diffracted by the slits at an angle ϑ - is given by

$$I(\vartheta) = I_0\left\{\frac{\sin\alpha}{\alpha}\cos\beta\right\}^2$$

when the slits are illuminated by a parallel beam of monochromatic light at normal incidence. Here, $\alpha=(ka/2)\sin\vartheta$, $\beta=(kd/2)\sin\vartheta$, $k=2\pi/\lambda$, and the constant I_0 depends upon the incident intensity. [4]



$$
\begin{aligned}
p(\vartheta) &= a_0\int_{-\frac{d}{2}-\frac{a}{2}}^{-\frac{d}{2}+\frac{a}{2}}\exp i\frac{2\pi x\sin\vartheta}{\lambda}\mathrm{d}x + a_0\int_{\frac{d}{2}-\frac{a}{2}}^{\frac{d}{2}+\frac{a}{2}}\exp i\frac{2\pi x\sin\vartheta}{\lambda}\mathrm{d}x \\
&= a_0\left(\exp -i\frac{2\pi\frac{d}{2}\sin\vartheta}{\lambda}+\exp i\frac{2\pi\frac{d}{2}\sin\vartheta}{\lambda}\right)\int_{-\frac{a}{2}}^{\frac{a}{2}}\exp i\frac{2\pi x'\sin\vartheta}{\lambda}\mathrm{d}x' \\
&= 2a_0\cos\frac{\pi d\sin\vartheta}{\lambda}\int_{-\frac{a}{2}}^{\frac{a}{2}}\exp i\frac{2\pi x'\sin\vartheta}{\lambda}\mathrm{d}x' \\
&= 2a_0\cos\frac{\pi d\sin\vartheta}{\lambda}\left[\frac{\exp i\frac{2\pi y'\sin\vartheta}{\lambda}}{i\frac{2\pi\sin\vartheta}{\lambda}}\right]_{-\frac{a}{2}}^{\frac{a}{2}} \\
&= 2a_0 a\cos\frac{\pi d\sin\vartheta}{\lambda}\frac{\sin\frac{\pi a\sin\vartheta}{\lambda}}{\frac{\pi a\sin\vartheta}{\lambda}} \equiv 2a_0 a\cos\beta\frac{\sin\alpha}{\alpha},
\end{aligned}
$$

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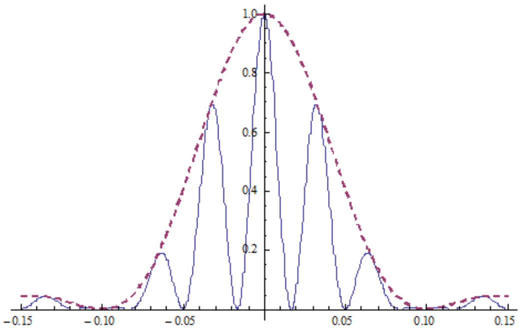
where we have written $x' \equiv x \pm d/2$. Since the intensity is proportional to the square of the amplitude, we have *[3]*

$$I(\vartheta) \propto \left\{ \frac{\sin\alpha}{\alpha} \cos\beta \right\}^2 .$$ *[1]*

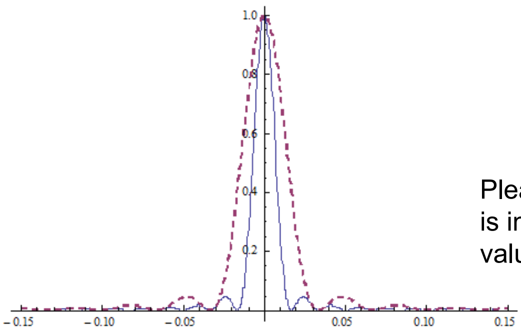
(d) Sketch the intensity distribution when $a = 10\ \mu\text{m}$, $\lambda = 1\ \mu\text{m}$, and $d \approx 3a$. Your diagram should be labelled to indicate the scale of important features. [4]



[4]

[2 marks for figure, 2 marks for labelling - not shown here: $I(\vartheta)$, ϑ (or $\sin\vartheta$), sinc function zeroes at $\sin\vartheta = n\lambda/a$, other fringe zeroes at $\sin\vartheta = (2n+1)\lambda/6a$.]

(e) Repeat your sketch for the special case $d = a$, and comment on the result. [4]



Please note that the x-axis here is incorrect if a=10um as above: values should be divided by 3

[2]

The case $d = a$ corresponds to a single slit of width $2a$. *[2]*

B4. (a) Explain what is meant by *dispersion*. Give examples of practical manifestations of dispersion, and of an application that exploits it. [5]

Dispersion describes the spreading of a wavepacket as it propagates, and corresponds to a variation in the phase velocity as a function of the frequency of sinusoidal components. In a dispersive system, the phase and group velocities will differ. *[2]*

Dispersion is apparent in the variation of refractive index with wavelength, apparent in rainbows, chromatic aberrations in imaging systems and the dispersion of a spectrum by a prism. Pulses from a laser will become longer as they propagate down an optical fibre (unless waveguide effects compensate), and short electrical pulses will be distorted by electrical transmission lines. Localized quantum particles appear to exhibit a range of momentum values. *[2]*

Dispersion by a prism can be used to produce an optical spectrum. The combination of dispersion with waveguide dispersion allows dispersion-free optical fibres to be designed. *[1]*

(b) Show that the wave equation

$$i\,m\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2}$$

where m is a constant, has complex exponential travelling wave solutions of the form

$$\psi(x,t) = \psi_0 \exp\left[i(kx - \omega t)\right].$$

Explain the significance of the parameters k and ω, and show that the *dispersion relation* between k and ω is given by $m\omega = k^2$. [5]

Substitution of the trial form into the wave equation gives

$$\begin{aligned} im\frac{\partial}{\partial t}\left(\psi_0 \exp i(kx-\omega t)\right) &= -\frac{\partial^2}{\partial x^2}\left(\psi_0 \exp i(kx-\omega t)\right) \\ \Rightarrow im(-i\omega)\psi_0 \exp\left[i(kx-\omega t)\right] &= k^2\psi_0 \exp\left[i(kx-\omega t)\right], \end{aligned}$$

which is true for all values of x and t (so the trial form is a solution) provided that *[2]*

$$m\omega = k^2. \qquad [\,1\,]$$

Here, k is the wavenumber - the number of radians of phase per unit distance along the direction of propagation of the periodic wave; ω is the angular frequency, or phase per unit time. *[2]*

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(c) What is meant by the *phase* and *group velocities*? Give, for the above example, an expression for the phase velocity v_p in terms of ω. [3]

The phase velocity is the velocity with which a wavefront of given displacement appears to propagate. The group velocity is the velocity with which the overall amplitude of a wavepacket – or the beat between two components – appears to propagate. *[2]*

When sinusoidal components are characterized by their frequency and wavenumber ω and k, the phase velocity $v_p = \omega/k$. Here, $m\omega = k^2$, so $v_p = k/m = \sqrt{m\omega}/m = \sqrt{\omega/m}$. *[1]*

(d) A travelling wave has two complex exponential components, equal in magnitude, with frequencies $\omega_0 \pm \delta\omega$ and wavenumbers $k_0 \pm \delta k$. Show that the wave may be written in the form

$$\psi(x,t) = \psi_1 \exp\left[i(k_0 x - \omega_0 t)\right] \cos(\delta k\, x - \delta\omega\, t)$$

and thus takes the form of a complex exponential travelling wave that is modulated by a slowly-varying, real periodic function. [4]

The travelling wave described will be

$$\psi(x,t) = \psi_0 \left(\exp\{i[(k_0 - \delta k)x - (\omega_0 - \delta\omega)t]\} + \exp\{i[(k_0 + \delta k)x - (\omega_0 + \delta\omega)t]\}\right) . \qquad [\,1\,]$$

A little rearrangement gives *[2]*

$$\begin{aligned} \psi(x,t) &= \psi_0 \exp\left[i(k_0 x - \omega_0 t)\right] \times \\ &\quad \left\{\exp\left[i\,(\delta k x - \delta\omega t)\right] + \exp\left[-i\,(\delta k x - \delta\omega t)\right]\right\} \\ &= 2\psi_0 \exp\left[i\,(k_0 x - \omega_0 t)\right] \cos\left(\delta k x - \delta\omega t\right) , \end{aligned}$$

which is of the required form, with $\psi_1 \equiv 2\psi_0$. *[1]*

(e) By considering how δk depends upon $\delta\omega$, show that the phase velocity of the wave differs from the group velocity of the modulating envelope by a factor of two. [3]

From the dispersion relation of part (b),

$$(k_0 \pm \delta k)^2 = m(\omega_0 \pm \delta\omega) . \qquad [\,1\,]$$

so

$$\begin{aligned} (k_0 + \delta k)^2 &= m(\omega_0 + \delta\omega) \\ (k_0 - \delta k)^2 &= m(\omega_0 - \delta\omega) \end{aligned}$$

thus

$$(k_0 + \delta k)^2 - (k_0 - \delta k)^2 = m(\omega_0 + \delta\omega) - m(\omega_0 - \delta\omega)$$ *[1]*

hence

$$4k_0\delta k = 2m\delta\omega.$$

The speed of propagation of the modulation, $\delta\omega/\delta k$, is hence $2k_0/m = 2\sqrt{\omega/m}$; this is the group velocity, and is twice the previously calculated phase velocity. *[1]*

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