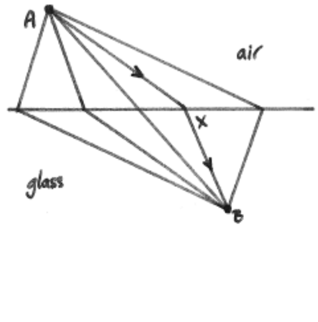
Wave Physics – exercise sheet 11.

(1 mark each; for illustration, some of the following answers are longer and more detailed than strictly necessary.)

1. A wave motion describes the propagation of a disturbance, whereby the motion at any given point depends upon that at its neighbouring points, subject to some retardation or finite speed of response. It therefore describes the bulk, collective motion of a medium, or of fields in space, and may be regarded as time-dependent field theory. Transverse waves include the motion of guitar strings, surface water waves, electromagnetic waves; longitudinal waves include sound, thermal waves (when regarded as the flow of thermal energy). Quantum (de Broglie) waves are neither transverse nor longitudinal, as are thermal waves if written in terms of temperature; other examples that are neither could include forms of Mexican waves, the wavefront of a chemical reaction, the wave disturbance across a comb of bees.
2. The displacement of a string causes the string to be curved, so that the action of the tension on any finite element no longer cancels. The net tension results in the acceleration of that element. The displacement of an element of an air column leads to a change in the volume occupied by a given element, and hence results in variations in pressure. A net force can therefore exist on any element, resulting in its acceleration.
3. Strings: mass per unit length, string tension (assuming small deflections, negligible resistance etc). Sound: density of medium, compressibility (Young's modulus etc). Shallow water waves: water density, acceleration due to gravity.
4. Travelling waves are those which maintain a constant form that is simply translated through space as time evolves. Standing waves maintain a spatially fixed form, that is multiplied by an evolving function of time. Standing waves may be written as superpositions of travelling waves, and vice-versa.
5. Sinusoidal waves render the mathematical analysis, in terms of differential equations, straightforward and, for linear systems, provide a complete basis set from which any solution can be formed as a superposition – even if the system shows dispersion. Their orthogonality allows the energy of a superposition to be easily determined by adding the contributions associated with the individual components. Complex exponential waves may be formed by superposing cosine and sine waves with complex coefficients; their real and imaginary parts hence correspond to real sinusoidal waves.
6. The energy density of a wave motion in a linear system is proportional to the square of the wave amplitude. For a guitar string, for example, the potential energy component is that for a stretched spring (which would be $\frac{1}{2}ky^2$), and the kinetic energy component is the usual $\frac{1}{2}\rho v^2$. Doubling the wave magnitude doubles both y and v. The wave intensity or power is the energy per unit time (and, if appropriate, per unit area) passing a given boundary, and is therefore given by the product of the energy density and the wave velocity.
7. Dispersion describes the spreading of a wave packet as it propagates, and corresponds to a variation in the phase velocity as a function of the frequency of sinusoidal components. The dispersion relations describe the relation between the frequency and wavenumber of sinusoidal waves, from which all other dispersive properties may be determined.
8. Phasors represent the magnitude and phase of a wave displacement. For a complex exponential wave, they correspond to the exponential function when plotted on an Argand diagram; for sinusoidal waves, one would use the complex number (with time-independent modulus) whose real part matches the wave displacement. The total disturbance of a superposition – typically resulting from interference or diffraction – is found by adding the wave amplitudes, or geometrically adding the phasors by joining them together head-to-tail.
9. The Huygens description allows the propagation of a wavefront to be determined by placing imaginary sources along a given wavefront and calculating the disturbance that would result some time later from those sources alone. When performed geometrically, the new wavefront lies along the common tangent to the circular wavefronts from adjacent contributions.
10. Parallel rays are focussed by a converging lens to single points in its focal plane, a focal length away from the lens. The positions of these points depend only upon the angle from which the parallel rays arrive. The arrival angle is thus mapped onto the displacement in the focal plane. For the position and angle to be interchanged, the initial and final points must both lie within focal planes. A diverging lens performs the same operation, except that the focal points are virtual.
11. Fermat's principle of least time is that in travelling between two points, a wave will follow the path that takes the least time.



In comparison with the straight-line route illustrated above, the path via X, although longer, involves spending less time in the slow medium (glass), the saving being greater than the increase in the time spent in air to get there.

12. Interference results when two wave motions are combined, giving a magnitude of wave disturbance that depends upon the component magnitudes and their relative phase. If the components arrive in step, their magnitudes are added, and constructive interference is said to occur; if they arrive out of step, then the resultant magnitude is reduced. When light is passed through a double slit, there are two possible routes (via the two slits) to any point in the plane of observation, and the relative phase of the corresponding contributions depends upon the position within the plane. The magnitude of the resultant disturbance hence varies with position, showing fringes as the interference varies periodically between constructive and destructive.

13. Diffraction is the interference that results from the partial and spatially-dependent obstruction of a wavefront. Fraunhofer diffraction is that which is observed in the image plane of the wave source, when the path length or phase for paths through a given point in the wavefront is a linear function of the coordinates describing that point. For plane-wave illumination of a mask, Fraunhofer diffraction is observed at distances that are large in comparison with the mask dimensions, or in the focal plane of a lens used to focus the transmitted light.

14. The continuity conditions describe the physical constraints relating the wave function (displacement and its derivatives) in the medium on one side of the interface to those on the other side at the interface. For a guitar string, for example, the continuity of the string and requirement for finite accelerations of even infinitesimal sections lead to the continuity of the wave displacement and its spatial derivative. For sound waves, the medium displacement must again be continuous, but the other condition is continuity of the wave pressure, so that it is the product of the derivative with the compressibility that must be continuous. A discontinuity in the material properties results in a change in wave speed and partial reflection of an incident wave.

15. Boundary conditions describe the physical constraints upon a wave motion imposed by an external system or structure. A guitar string, for example, is constrained to fixed points at the bridge and fret; the air column within an organ pipe is fixed in displacement where it meets the wall of the pipe, and in pressure to that of the atmosphere where it is open. Sinusoidal wave motions therefore have nodes or anti-nodes of displacement (anti-nodes and nodes of pressure variation) at the closed and open ends of organ pipes. Depending upon the instrument geometry, it may support all harmonics of the fundamental frequency, or may be limited to only odd-numbered harmonics.

16. A wave equation is linear when the displacement y appears only as linear functions (y, $\mathrm{d}y/\mathrm{d}x$ etc, rather than y^2, $y\,\mathrm{d}y/\mathrm{d}x$ and the like). If the wave equation is linear, then superpositions of its solutions will also themselves be solutions of the wave equation.

17. The phase velocity is the velocity with which a wavefront of given displacement appears to propagate. The group velocity is the velocity with which the overall amplitude of a wavepacket – or the beat between two components - appears to propagate. When sinusoidal components are characterized by their frequency and wavenumber ω and k_ω, the phase velocity $v_\mathrm{p} = \omega/k$ and the group velocity $v_\mathrm{g} = \mathrm{d}\omega/\mathrm{d}k_\omega$. The dispersion relations link ω to k_ω, and hence allow the phase and group velocity to be determined.

18. The frequency spectrum indicates the respective strengths of components of various frequencies in a time-dependent signal or wave motion, where a single component will be a pure sinusoidal wave. Examples include the image recorded on the plate or detector array of a spectrograph or spectrometer, the musical notes on a stave, or the graphical output of a spectrum analyser. Single frequencies appear as δ-functions (usually smoothed to some extent by the finite response of the instrument), and harmonics as a comb of equally-spaced peaks or lines.

19. The principle of Fourier synthesis is that any wavefunction can be built up from sinusoidal wave components of appropriate magnitudes and phases. The corresponding principle of Fourier analysis is that any wavefunction can be broken down into these components. If we know how sinusoidal wave components behave in a particular system, we can hence determine the behaviour of an arbitrary wave motion by breaking it into sinoisoidal components, allowing for their known behaviour, and recombining them into the composite wave.

20. The Fourier transform allows a function of time or position to be instead represented as a function of frequency or spatial frequency – i.e. by the spectrum of sinusoidal or complex exponential components into which it may be resolves. The component with a given frequency is obtained by multiplying the function by a sinusoidal (or complex exponential) wave with the same (or negative) frequency, and integrating over the range of the function, for example

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \exp(-i\omega t)\,\mathrm{d}t \tag{11.1}$$

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-ikx)\,\mathrm{d}x \tag{11.2}$$

[The leading factor of $1/\sqrt{2\pi}$ is to some extent an arbitrary choice, determined by whether the aim is to symmetrize the Fourier transform and its inverse or to normalize the spectral intensity.]

21. Convolution (Lat. rolling together) can be thought of as a blurring process, modifying one function by spreading each point over an adjacent range (known in optics as the point-spread-function). To obtain the convolution of two functions $A(x)$ and $B(x)$, we take one of the functions (say $A(x)$) and, for each point x_0, place the function $A(x_0)\ B(x+x_0)$ centred on it and scaled according to $A(x_0)$. The convolution is the sum of all such functions as x_0 ranges from $-\infty$ to ∞. The convolution theorem states that the Fourier transform of the convolution of two functions is equal to the product of the Fourier transforms of the individual functions (and the FT of the product is equal to the convolution of the individual transforms). Since the spectrum and diffraction pattern correspond to the Fourier transforms of time and position-dependent functions, their evaluation may be simplified if the original function can be broken down into products and convolutions of basic functions.

22. An operator is a recipe by which particular parameters may be obtained from the wavefunction at a given point and time. For sinusoidal wave motions, for example, the frequency and wavenumber are given by (depending upon the chosen sign convention) $\pm$i d/dt and $\pm$(1/i) d/dx (where the recipe here requires the result to be normalized to the value of the wave motion at each point). If the wave has a single frequency or wavenumber, then the recipe will give the same value of frequency or wavenumber at all points and times.

23. If a wave motion contains components with a range of frequencies, then the mean frequency may be obtained from the spectrum (Fourier transform), by calculating its weighted average, or – using operators – from the time-dependent wave function by calculating the expectation value. The weighting is proportional to the wave intensity – i.e., to the square of the wave displacement. The standard deviation may be determined using the usual statistical method from the spectrum (which is just a distribution as a function of frequency). By applying the frequency operator twice, however, to give an operator for ω^2, the standard deviation (or uncertainty) may also be found from the time-domain function, as the difference between $<\omega^2>$ and $<\omega>^2$.

24. The bandwidth theorem is that if we wish to limit the extent of a wavepacket in one dimension (space, time, frequency, wavenumber), then it will span at least a certain range of the space of the conjugate variable. A brief pulse therefore comprises a wide range of frequencies. If we express the spread or range of the wavepacket in the space of a given variable by the uncertainty in the variable, then the bandwidth theorem is that the product of the uncertainties cannot fall below a given value. For example

$$\Delta_x \Delta_k \geq \frac{1}{2} \tag{11.3}$$

$$\Delta_t \Delta_\omega \geq \frac{1}{2} \tag{11.4}$$

In classical systems, the bandwidth theorem hence means that to discriminate between similar frequencies there is a minimum duration for the measurement, and that for an instrument to respond to brief pulses it must have at least a certain bandwidth. In quantum systems, the bandwidth theorem is equivalent to Heisenburg's uncertainty principle, that one cannot simultaneously know, for example, the position and momentum of an electron or photon, with arbitrary precision.

25. The Doppler effect is the shift in frequency recorded when there is a relative motion between the observer and the wave source. The classical derivation of the Doppler effect considers the progression of propagating wavefronts relative to the motion of the source and observer, and establishes the times and positions when the wavefronts and source or observer meet. The Doppler shift also emerges automatically from the transformations of special relativity, and from quantum-mechanical treatments, in which waves have quantized energies and momenta, from the principles of conservation of energy and momentum.