SEMESTER 2 EXAMINATION 2015-2016

CLASSICAL MECHANICS

Duration: 120 MINS (2 hours)

This paper contains 9 questions.

Answers to Section A and Section B must be in separate answer books

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language Word to Word® translation dictionary (paper version) is permitted provided it contains no notes, additions or annotations.

Section A

A1. Explain what are meant by the *centre of mass* and *moment of inertia*. Give expressions for both properties, for a system comprising a number of particles with masses m_i and positions $\mathbf{r}_i$. [4]

A2. A uniform solid disc of radius a has a hole of radius b bored axially through its centre to form a ring of mass M. Show that the moment of inertia I of the ring about its axis of rotational symmetry is

$$I = \frac{1}{2}M\left(a^2 + b^2\right).$$ [4]

A3. A cricketer strikes a ball with a bat of mass m and moment of inertia I (about its centre of mass), and imparts an impulse Δp. Show that, if no impulse is to be felt at the handle, the change in the angular velocity of the bat must be $\Delta\omega = \Delta p/(mD)$, where D is the distance from the handle to the bat's centre of mass. [2]

Hence show that the ball should strike the bat a distance $D + I/(mD)$ from the handle – a point known as the *centre of percussion*. [2]

[Assume the angular velocity and moment of inertia to be about the same axis, perpendicular to the plane defined by the handle, point of impact and impulse.]

A4. The magnitude g of the acceleration due to gravity is found to be greater down a mine than it is on the Earth's surface. Show that this can be explained, taking the Earth to have spherical symmetry, if

$$\rho_{\rm s} < \frac{2}{3}\rho_{\rm av}$$

where $\rho_{\rm av}$ is the average density of the Earth and $\rho_{\rm s}$ its density at the surface. [4]

A5. Show that, if a spacecraft of total mass $m(t)$ propels itself by ejecting exhaust gas from its rocket motor with a relative velocity $\mathbf{u}$, then its velocity $\mathbf{v}(t)$ satisfies

$$m\mathrm{d}\mathbf{v} = -\mathbf{u}\mathrm{d}m,$$ [2]

and hence, making clear any assumptions in your derivation, that the initial and final speeds v_i and v_f are related to the initial and final masses m_i and m_f by

$$v_f = v_i + u \ln \frac{m_i}{m_f}.$$ [2]

TURN OVER

Section B

B1. Show that, if a fixed-length vector $\mathbf{A}$ rotates with angular velocity ω about an axis defined by the vector $\hat{\boldsymbol{\omega}}$, and we define $\boldsymbol{\omega} \equiv \omega\hat{\boldsymbol{\omega}}$, then

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \boldsymbol{\omega} \times \mathbf{A}.$$ [4]

The unit vectors $\hat{\mathbf{i}}'$, $\hat{\mathbf{j}}'$ and $\hat{\mathbf{k}}'$ of a rotating coordinate frame rotate with angular velocity ω about an axis $\hat{\boldsymbol{\omega}}$, so that a vector $\mathbf{a} \equiv a_i\hat{\mathbf{i}} + a_j\hat{\mathbf{j}} + a_k\hat{\mathbf{k}}$ in an inertial frame $\{\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}\}$ may be written at a given time as $\mathbf{b} \equiv b_i\hat{\mathbf{i}}' + b_j\hat{\mathbf{j}}' + b_k\hat{\mathbf{k}}'$. Show that

$$\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} = \dot{\mathbf{b}} + \boldsymbol{\omega} \times \mathbf{b}$$

and hence that

$$\frac{\mathrm{d}^2\mathbf{a}}{\mathrm{d}t^2} = \ddot{\mathbf{b}} + 2\boldsymbol{\omega} \times \dot{\mathbf{b}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}),$$

where $\dot{\mathbf{b}} \equiv \dot{b}_i\hat{\mathbf{i}}' + \dot{b}_j\hat{\mathbf{j}}' + \dot{b}_k\hat{\mathbf{k}}'$, $\ddot{\mathbf{b}} \equiv \ddot{b}_i\hat{\mathbf{i}}' + \ddot{b}_j\hat{\mathbf{j}}' + \ddot{b}_k\hat{\mathbf{k}}'$, and $\dot{b}_i \equiv \mathrm{d}b_i/\mathrm{d}t$ etc. [6]

Hence show that, for a particle of mass m subject to gravitational acceleration $\mathbf{g}$ and an applied force $\mathbf{F}$, the equation of motion in the rotating frame will be

$$m\ddot{\mathbf{b}} = \mathbf{F} + m\mathbf{g} - \underbrace{2m\boldsymbol{\omega} \times \dot{\mathbf{b}}}_{*} - \underbrace{m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})}_{**}.$$ [2]

Explain the significance of the terms marked * and **, and why that marked ** may generally be neglected when the axes are referred to the Earth's surface. [3]

Show that if wind is to flow undeflected over the surface of the Earth, then there must be a sideways pressure gradient given by the geostrophic wind condition

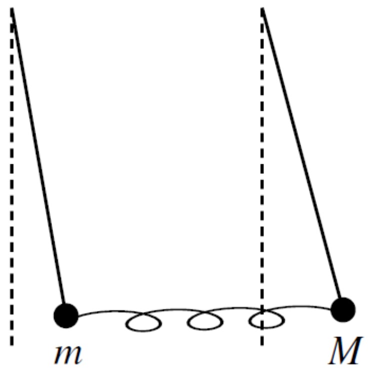
$$\frac{\mathrm{d}P}{\mathrm{d}y} = 2\rho\omega\sin(\alpha)\,v$$

where ρ is the air density, v the wind speed, α the latitude and ω the angular velocity at which the Earth rotates. [3]

Calculate the horizontal distance over which a pressure drop of 400 Pa is required for a geostrophic wind speed of 13 m s^{-1} at a latitude of 50° N at sea level. The density of air at sea level may be taken to be 1.225 kg m^{-3}. [2]

B2. (a) Explain what is meant by the *normal mode* of an oscillating system. [2]

(b) Two simple pendulums, each of length l, have bobs of masses m and M. The pendulums are coupled by a weak spring of spring constant k, as shown in the diagram below.



If the displacements of the pendulum bobs are x and X, show that, for small displacements, the motion of the system may be described by

$$\frac{d^2}{dt^2}\begin{pmatrix} x \\ X \end{pmatrix} = \begin{pmatrix} -\frac{g}{l} - \frac{k}{m} & \frac{k}{m} \\ \frac{k}{M} & -\frac{g}{l} - \frac{k}{M} \end{pmatrix} \begin{pmatrix} x \\ X \end{pmatrix}.$$ [6]

(c) Consider the normal modes of this system and find

(i) the corresponding eigenfrequencies; and [5]

(ii) the corresponding eigenvectors. [3]

(d) The system is released from rest with m in its equilibrium position and M displaced a small distance d directly away from the other pendulum. Find an expression describing the subsequent motion. [4]

TURN OVER

B3. State the relationship between torque and angular momentum, and explain what is meant by *precession* in the context of rotational motion. Give an example of precession, and state the physical principle from which it results. [6]

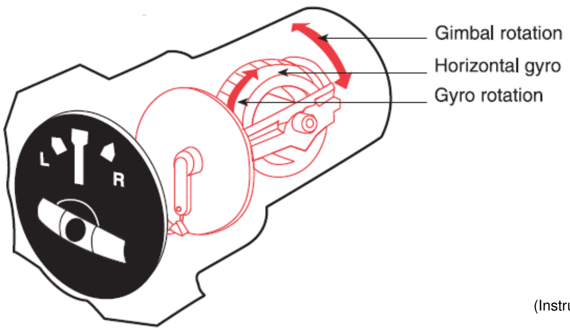
A disc of mass M spins with angular velocity ω about a light axle along its axis of rotational symmetry, about which the disc has a moment of inertia I. The combination is suspended by attaching the axle, at a distance a from the disc's centre of mass, to a rigid support, and the axle assumes a constant angle α to the vertical. Show that the moment of the disc's weight about the support is

$$Mg\,a\sin\alpha$$

and that the spinning disc precesses about the support with angular frequency

$$\Omega = \frac{Mg\,a}{I\omega}.$$ [4]

An aircraft's rate-of-turn indicator comprises a gyroscope whose rotation axis during straight flight lies athwart the aircraft (i.e. horizontally from left to right), as shown below. The gyroscope spins within a spring-loaded gimbal that allows it to rotate about the longitudinal (fore-aft) axis of the aircraft, and a pointer linked to the gimbal indicates this rotation. If the aircraft heading changes, the gyroscope exerts a torque that balances the restoring torque from the spring at an angle proportional to the rate of change of heading.



(Instrument Flying Handbook FAA-H-8083-15)

If the gyroscope spins at 3,000 rpm, its moment of inertia is $1.7 \times 10^{-5}\,\mathrm{kg\,m^2}$, and the spring constant at the pointer is $10^{-3}\,\mathrm{N\,m\,rad^{-1}}$, find the angle through which the pointer turns if the aircraft makes one turn every two minutes. [6]

Explain how the instrument will be affected by rotation about (a) a lateral axis when the aircraft changes pitch and (b) a longitudinal axis when it rolls. [4]

B4. A satellite orbits the Earth, subject only to the Earth's gravitational attraction. Explain why the vector angular momentum is conserved and why this means that the orbit lies in a plane. What other quantity or quantities is/are conserved? [5]

A spacecraft is moving in a circular, geostationary orbit of radius $6.6\, r_{\mathrm{E}}$ around the Earth, where r_{E} is the Earth's radius. A brief impulse from the spacecraft's rocket motors changes its direction of motion through an angle α towards the Earth, without any change in speed. At its closest approach, the new orbit is $4.2\, r_{\mathrm{E}}$ from the centre of the Earth.

Sketch the initial and final orbits, and indicate the spacecraft's velocities just before and after the rocket impulse. [2]

Find the angle α through which the spacecraft is deflected. [9]

Find the maximum distance of the spacecraft from the Earth's centre, and the period of the new orbit. [4]

[The Earth may be taken to be spherically symmetric, and its atmosphere and reduced mass effects may be neglected.]

END OF PAPER