
SEMESTER 2 EXAMINATION 2016-2017

CLASSICAL MECHANICS

Duration: 120 MINS (2 hours)

This paper contains 9 questions.

Answers to Section A and Section B must be in separate answer books.

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language dictionary is permitted ONLY IF it is a paper version of a direct ‘Word to Word’ translation dictionary AND it contains no notes, additions or annotations.

6 page examination paper.

Section A

A1. Show that, if a fixed-length vector $\mathbf{A}$ rotates with angular velocity ω about an axis defined by the vector $\hat{\boldsymbol{\omega}}$, and we define $\boldsymbol{\omega} \equiv \omega\hat{\boldsymbol{\omega}}$, then

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \boldsymbol{\omega} \times \mathbf{A}.$$ [4]

A2. State Kepler's laws and outline the physical assumptions upon which they are based. [4]

A3. Show, either by adapting Gauss's law or by direct integration, that the gravitational attraction exerted at a point $\mathbf{r}$ by a spherically symmetrical mass distribution is the same as that exerted by a point mass, positioned at the centre of spherical symmetry $\mathbf{R}$, whose mass is equal to that of the part of the distribution that lies within a distance $|\mathbf{r} - \mathbf{R}|$ of $\mathbf{R}$. [4]

A4. Show, using the *perpendicular axis theorem* or otherwise, that the moment of inertia I of a square plate, about a perpendicular axis through its centre, is given by

$$I = \frac{MA}{6}$$

where M is the mass of the plate and A its area. [4]

A5. At a shooting range in central Australia (27° S), a rifle bullet is fired with an initial speed (*muzzle velocity*) v horizontally towards the west. Explain how, and in which direction, it is deflected as a result of the Earth's rotation. [4]

Section B

B1. A bricklayer wishes to clear some left-over bricks from the top of a building, and rigs up a simple hoist by attaching an empty barrel of mass $m_1 = 10$ kg to a line that passes over a pulley at the top. He hoists the barrel to the top of the building, a height $h = 25$ m above the ground, and secures the line at the bottom. He then fills the barrel with bricks of total mass $m_2 = 90$ kg.

(a) Calculate (i) the tension in the line, and (ii) the load borne by the pulley. [4]

The bricklayer, of mass $m_3 = 80$ kg, then goes to the bottom and casts off the line. Unfortunately the barrel starts moving down, jerking him off the ground.

He decides to hang on.

(b) (i) Show that the acceleration a experienced by the barrel will be

$$a = g\,[1 - 2m_3/\,(m_1 + m_2 + m_3)]\,,$$

and hence (ii) calculate the load now borne by the pulley. [7]

Halfway up, the bricklayer meets the barrel coming down.

(c) Neglecting friction and air resistance, calculate the relative speed with which the barrel strikes the bricklayer. [3]

The bricklayer continues to be pulled to the top, banging his head against the pulley beam. The barrel hits the ground and bursts at its bottom, allowing all the bricks to spill out, leaving just the barrel attached to the line.

(d) Assuming that the barrel was slowed to a halt when it struck the bricklayer, find the kinetic energy of the brick-laden barrel just before it hits the ground. [2]

The bricklayer is now heavier than the barrel, and so starts down again at high speed. Halfway down, he meets the barrel coming up.

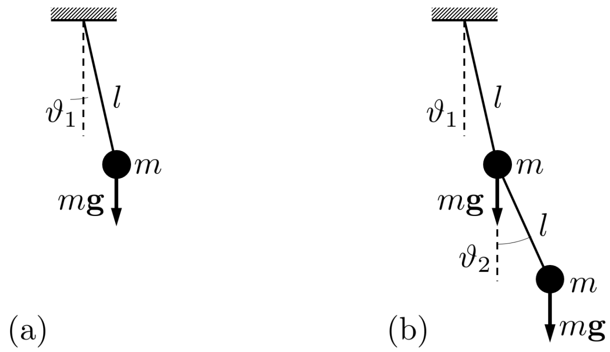
(e) Calculate the relative speed of the barrel when it strikes the bricklayer. [2]

The bricklayer continues to fall, and lands uncomfortably on the bricks. At this point, he loses his presence of mind, and lets go of the line.

(f) Calculate the kinetic energy of the empty barrel when it lands on the bricklayer. [2]

TURN OVER

B2. (a) Explain what is meant by (i) *simple harmonic motion* and (ii) the *normal mode* of an oscillating system. [4]



(b) A simple pendulum of length l has a bob of mass m, subject to gravitational acceleration g. The position of the pendulum bob is described by the angle ϑ_1 between the pendulum and the vertical, as shown in figure (a) above. Show, explaining any approximations made, that for small angles $|\vartheta_1| \ll 1$, the motion will be governed by the equation

$$\frac{d^2\vartheta_1}{dt^2} = -\frac{g}{l}\vartheta_1.$$ [4]

(c) A second pendulum, also of length l and with a bob of mass m, is suspended from the bob of the first pendulum, and its position described by its angle ϑ_2 to the vertical, as shown in figure (b) above. Show, again for small angles $|\vartheta_{1,2}| \ll 1$, that

$$\frac{d^2\vartheta_1}{dt^2} = \frac{g}{l}\left(\vartheta_2 - 2\vartheta_1\right).$$ [3]

(d) Show that, for small angles $|\vartheta_{1,2}| \ll 1$, the horizontal displacement of the second bob is approximately $l\,(\vartheta_1 + \vartheta_2)$ and hence show that

$$\frac{d^2\vartheta_1}{dt^2} + \frac{d^2\vartheta_2}{dt^2} = -\frac{g}{l}\vartheta_2.$$ [3]

(e) Show that the motion of the double pendulum is therefore governed by the equation

$$\frac{d^2}{dt^2}\begin{pmatrix} \vartheta_1 \\ \vartheta_2 \end{pmatrix} = \begin{pmatrix} -\frac{2g}{l} & \frac{g}{l} \\ \frac{2g}{l} & -\frac{2g}{l} \end{pmatrix}\begin{pmatrix} \vartheta_1 \\ \vartheta_2 \end{pmatrix}.$$ [2]

(f) Hence derive the frequencies of the modes of the double pendulum. [4]

B3. A comet of mass m moves in the gravitational field of a star of mass M, and its position is described by its polar coordinates (r, ϑ) relative to the star. The gravitational potential is given by $\mathcal{V}(r) = GMm/r$. Assume that $M \gg m$.

(a) Show that the angular momentum of the comet about the star will be $L = mr^2\dot{\vartheta}$, where $\dot{\vartheta}$ signifies $\mathrm{d}\vartheta/\mathrm{d}t$, the rate of change of ϑ with time. [2]

(b) Show that the comet's total energy $\mathcal{E}$ may be written as

$$\mathcal{E} = \frac{m}{2}\dot{r}^2 + \left(\frac{L^2}{2mr^2} - \frac{GMm}{r}\right) \equiv \frac{m}{2}\dot{r}^2 + \mathcal{U}(r),$$

where $\dot{r} \equiv \mathrm{d}r/\mathrm{d}t$ and $\mathcal{U}(r)$ is the effective potential in which the comet's radial motion occurs. [4]

(c) Assuming that the comet follows an elliptical orbit with the star at one focus, show from these results that the length $2a$ of the ellipse's major axis will be

$$2a = \frac{GMm}{-\mathcal{E}}.$$ [4]

(d) By differentiating the total energy with respect to the time t, derive the equation of radial motion of the comet,

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{L^2}{m^2 r^3} - \frac{GM}{r^2}.$$ [2]

(e) By writing $\frac{\mathrm{d}}{\mathrm{d}t} \equiv \dot{\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta} \equiv \frac{L}{mr^2}\frac{\mathrm{d}}{\mathrm{d}\vartheta}$ and making the substitution $r \equiv 1/u$, show that the equation of motion may be rewritten as

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\vartheta^2} = -u + \frac{GMm^2}{L^2}.$$ [4]

(f) Hence show that the comet will trace out a path $r(\vartheta)$ of the form

$$r = \frac{L^2}{GMm^2\,(1 + \alpha\cos\vartheta)}$$

where

$$\alpha^2 = 1 + \frac{2L^2\mathcal{E}}{(GMm)^2\,m}.$$ [4]

TURN OVER

B4. (a) State the relationship between torque and angular momentum, and explain what is meant by *precession* in the context of rotational motion. Give an example of precession, and state the physical principle from which it results. [5]

A spinning top comprises a ring of mass M that is connected by spokes of negligible mass to a light axle along the axis of rotational symmetry, about which the top has a moment of inertia I. The axle is suspended at a pivot a distance a from the ring's centre of mass. When the top spins with angular velocity ω the axle assumes a constant angle α to the vertical.

(b) Draw a suitably labelled diagram illustrating this situation. [2]

(c) Show that the moment of the disc's weight about the support is

$$Mg\,a\sin\alpha$$

and hence that, if α is constant, the spinning disc precesses about the support with angular frequency

$$\Omega = \frac{Mg\,a}{I\omega}.$$ [4]

(d) The ring, of uniform thickness d, has an inner radius p and an outer radius q. Show that, in terms of its total mass M, its moment of inertia is

$$I = \frac{1}{2}M\left(p^2 + q^2\right).$$ [4]

(e) Hence find the time $T = 2\pi/\Omega$ it takes for the top to precess around a vertical axis if $p = 8$ cm, $q = 10$ cm, $d = 1$ cm, $a = 0.1$ m, $M = 0.3$ kg and the top spins at 5 revolutions per second. [2]

(f) The top is temporarily prevented from precessing by blocking the path of the axle, while the top continues to rotate at an angle α to the vertical with angular frequency ω. The top is then released. Describe its subsequent motion. [3]

END OF PAPER