SEMESTER 2 EXAMINATION 2018-2019

CLASSICAL MECHANICS

Duration: 120 MINS (2 hours)

This paper contains 9 questions.

Answers to Section A and Section B must be in separate answer books

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language dictionary is permitted ONLY IF it is a paper version of a direct 'Word to Word' translation dictionary AND it contains no notes, additions or annotations.

Section A

A1. State the parallel and perpendicular axis theorems, explaining clearly the situations to which they apply and any terms or symbols used. [4]

A2. State Kepler's laws and outline the physical assumptions upon which they are based. [4]

A3. Define torque, show how it may be expressed as a vector product, and state how it affects the angular momentum of a body upon which it acts. [2]

Hence show that if a particle orbiting a point is subject only to a central force directed towards that point, its angular momentum is conserved, irrespective of the shape of the orbit. [2]

A4. Give a formula for the *reduced mass* of a system of two particles of masses m_1 and m_2. For what situations is it a useful quantity? [2]

Calculate the reduced mass for the Earth-Moon system, given that the masses of the Earth and Moon are 5.972×10^{24} kg and 7.346×10^{22} kg respectively. [2]

A5. Explain the physical principles behind *Buys Ballot's law*: that, if you stand with your back to the wind in the northern hemisphere, the atmospheric pressure will be lower on your left than to your right. [4]

Section B

B1. (a) Explain what is meant by a *normal mode* of an oscillating system. [2]

A bead of mass m slides freely on a smooth circular ring of negligible mass and radius R, which is attached at its centre to a small hub of mass m. The ring is free to rotate in its own plane about a fixed axis through a point on its circumference, and the system makes *small* oscillations under gravity.

(b) Sketch the system described. [2]

(c) For the ring position shown in your diagram, indicate the point about which the bead rotates. [1]

(d) Derive the matrix equation governing the angular coordinates ϑ and φ of the ring relative to the pivot, and bead relative to the ring, respectively,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{pmatrix}\vartheta\\ \varphi\end{pmatrix} = \frac{g}{R}\begin{pmatrix}-2 & 1\\ 2 & -2\end{pmatrix}\begin{pmatrix}\vartheta\\ \varphi\end{pmatrix}.$$ [6]

(e) Hence derive the eigenvalue matrix equation and solve it to determine the angular frequencies of the two normal modes of the system. [5]

(f) Calculate the (unnormalized) eigenvectors associated with each eigenvalue. [2]

(g) Describe the motions of the ring and bead in each mode. [2]

TURN OVER

B2. (a) Explain what is meant by an object's *moment of inertia* about an axis, and define it mathematically in terms of the distribution of the object's mass. [2]

(b) Show that the moment of inertia of a uniform solid cylinder of radius R, length L and mass M is given by $I = \frac{1}{2}MR^2$. [4]

The *'bouncing bomb'* used in the *'Dam Buster'* raids of *Operation Chastise* was a cylinder measuring 1.27 m in diameter by 1.52 m in length, with a mass of 4200 kg. It was carried beneath the bomber aircraft with the axis of the cylinder parallel to the wings, perpendicular to the direction of travel. To stabilize the bomb once deployed, it was rotated whilst aboard the aircraft at 500 revolutions per minute, the sense of rotation being clockwise when viewed from the left of the aircraft. The aircraft approached its target at a speed of 95 m s^{-1}.

(c) Sketch the situation described. [2]

(d) Calculate the speed relative to the ground of the lowest point of the bomb. [1]

(e) Calculate the magnitude and direction of the bomb's angular momentum. [3]

To line up for the approach to the target, the bomber aircraft needed to turn to the right. The pilot therefore lowered the right wing, so that the aircraft initially rotated about a longitudinal (fore-aft) axis at a rate of 10 deg s^{-1}.

(f) Calculate the magnitude and direction of the gyroscopic torque exerted upon the aircraft by the rotating bomb. [4]

(g) What effect will this torque have had upon the aircraft? [2]

(h) How large a force would need to be applied to the tail of the aircraft to counteract this effect? You may assume the tail to be about 11 m from the centre of mass of the aircraft. [1]

(i) In which direction should this force be applied? [1]

B3. (a) State the relationship between the acceleration g due to gravity at the Earth's surface, Newton's gravitational constant, and the mass and radius of the Earth (assuming it to be spherically symmetric). Define any symbols used. [2]

A Galileo GNSS satellite is launched by rocket into orbit around the Earth. After the first boost stage, the rocket is 1130 km above the Earth's surface and has a velocity of 9.2 km s^{-1} perpendicular to a line from the Earth's centre.

(b) Sketch the situation described, and the orbit established after the first boost stage. [3]

(c) Show, by considering two conservation laws, that the furthest distance of the satellite from the Earth's centre during the subsequent orbital motion may be written as

$$r_a = \frac{r_p}{\frac{2GM}{r_p v_p^2} - 1},$$

where G is the gravitational constant, M the mass of the Earth, and r_p and v_p are respectively the radial distance from the centre of the Earth and the rocket's velocity immediately after the first boost stage. [8]

(d) By expressing GM in terms of the Earth's radius r_E and the gravitational acceleration g at its surface, show that the furthest distance of the satellite may be written as

$$r_a = \frac{r_p}{\frac{2g\, r_E^2}{r_p v_p^2} - 1}.$$ [1]

(e) Calculate the numerical value of this distance. [1]

(f) Derive the value of the eccentricity of the orbit. [5]

The Earth's radius may be taken to be 6370 km, and gravitational acceleration at the Earth's surface to be 9.81 m s^{-1}.

TURN OVER

B4. (a) Show that, if a fixed-length vector $\mathbf{A}$ rotates with angular velocity ω about an axis defined by the vector $\hat{\omega}$, and we define $\boldsymbol{\omega} \equiv \omega\hat{\omega}$, then

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \boldsymbol{\omega} \times \mathbf{A}.$$ [4]

The unit vectors $\hat{\mathbf{i}}'$, $\hat{\mathbf{j}}'$ and $\hat{\mathbf{k}}'$ of a rotating coordinate frame rotate with angular velocity ω about an axis $\hat{\omega}$, so that a vector $\mathbf{a} \equiv a_i\hat{\mathbf{i}} + a_j\hat{\mathbf{j}} + a_k\hat{\mathbf{k}}$ in an inertial frame $\{\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}\}$ may be written at a given time as $\mathbf{b} \equiv b_i\hat{\mathbf{i}}' + b_j\hat{\mathbf{j}}' + b_k\hat{\mathbf{k}}'$.

(b) Show that

$$\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} = \dot{\mathbf{b}} + \boldsymbol{\omega} \times \mathbf{b}$$

and hence that

$$\frac{\mathrm{d}^2\mathbf{a}}{\mathrm{d}t^2} = \ddot{\mathbf{b}} + 2\boldsymbol{\omega} \times \dot{\mathbf{b}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}),$$

where $\dot{\mathbf{b}} \equiv \dot{b}_i\hat{\mathbf{i}}' + \dot{b}_j\hat{\mathbf{j}}' + \dot{b}_k\hat{\mathbf{k}}'$, $\ddot{\mathbf{b}} \equiv \ddot{b}_i\hat{\mathbf{i}}' + \ddot{b}_j\hat{\mathbf{j}}' + \ddot{b}_k\hat{\mathbf{k}}'$, and $\dot{b}_i \equiv \mathrm{d}b_i/\mathrm{d}t$ etc. [6]

(c) Hence show that, for a particle of mass m subject to gravitational acceleration $\mathbf{g}$ and an applied force $\mathbf{F}$, the equation of motion in the rotating frame will be

$$m\ddot{\mathbf{b}} = \mathbf{F} + m\mathbf{g} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) - 2m\boldsymbol{\omega} \times \dot{\mathbf{b}}.$$ [2]

The *vibrating structure gyroscope* comprises a miniature tuning fork, which may be taken to lie in the $x' - z'$ plane with the z'-axis following the axis of symmetry. The prongs of the tuning fork are driven in opposite directions in the x' direction so that their displacements at time t are $x' = \pm x'_0 \sin(\omega_0 t)$.

(d) Show that, if the gyroscope rotates about an angular velocity vector $\mathbf{\Omega}$ with components $\Omega_{x'}$, $\Omega_{y'}$, $\Omega_{z'}$, then the prongs will experience Coriolis forces

$$\mathbf{F}_{Cor} = \mp 2m x'_0 \omega_0 \cos(\omega_0 t)\, \mathbf{\Omega} \times \hat{\mathbf{i}}',$$

where m is the effective mass of each prong. [3]

(e) Hence show that in the y' direction, the relative displacement of the prongs, neglecting any resonance effects, will be

$$\Delta y' = \frac{4x_0'}{\omega_0}\Omega_{z'} \cos(\omega_0 t). \qquad [\,3\,]$$

(f) If the device is driven at a frequency of 15 kHz with amplitude 5 μm, find the amplitude of the relative motion when the device is used to measure the rotation of a Formula 1 engine at 10,000 rpm (revolutions per minute). [2]

END OF PAPER