SEMESTER 2 EXAMINATION 2017-2018

CLASSICAL MECHANICS

Duration: 120 MINS (2 hours)

This paper contains 9 questions.

Answers to Section A and Section B must be in separate answer books

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language dictionary is permitted ONLY IF it is a paper version of a direct ‘Word to Word’ translation dictionary AND it contains no notes, additions or annotations.

Section A

A1. Define the centre of mass coordinate $\mathbf{R}$ for a collection of particles with masses m_i, and show that the total momentum $\mathbf{P} = M\dot{\mathbf{R}}$, where M is the total mass. [4]

The centre of mass coordinate is defined for N particles with masses m_i and positions $\mathbf{r}_i$ as

$$\mathbf{R} = \frac{1}{M}\sum_{i=1}^{N} m_i \mathbf{r}_i \qquad [\,1\,]$$

where the total mass of the collection of particles is

$$M = \sum_{i=1}^{N} m_i. \qquad [\,1\,]$$

Hence, differentiating the above expression assuming the masses to remain constant,

$$\dot{\mathbf{R}} = \frac{1}{M}\sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i = \frac{1}{M}\mathbf{P} \qquad [\,1\,]$$

where the total momentum is

$$\mathbf{P} = \sum_{i=1}^{N} \mathbf{p}_i = \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i. \qquad [\,0.5\,]$$

Thus

$$\mathbf{P} = M\dot{\mathbf{R}}. \qquad [\,0.5\,]$$

A2. State the *centre of mass* condition for an isolated system of particles with masses m_i and positions $\mathbf{r}_i = \boldsymbol{\rho}_i + \mathbf{R}$, where $\boldsymbol{\rho}_i$ are the positions relative to the centre of mass coordinate $\mathbf{R}$. [1]

The centre of mass condition is that

$$\sum_i m_i \boldsymbol{\rho}_i = 0. \qquad [\,1\,]$$

[Alternatively, in words: the mass-weighted mean position relative to the centre of mass is zero.]

Hence show that the total kinetic energy T of the particles may be written as

$$T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\sum_i m_i \dot{\boldsymbol{\rho}}_i^2,$$

where $M = \sum_i m_i$ is the combined mass of the system. [3]

The total kinetic energy may be written in terms of the positions relative to the centre of mass as

$$\begin{aligned} T &= \frac{1}{2}\sum_i m_i\left(\dot{\mathbf{R}}+\dot{\boldsymbol{\rho}}_i\right)^2 = \frac{1}{2}\sum_i m_i\left(\dot{\mathbf{R}}^2+2\dot{\mathbf{R}}\cdot\dot{\boldsymbol{\rho}}_i+\dot{\boldsymbol{\rho}}_i^2\right) \\ &= \frac{1}{2}M\dot{\mathbf{R}}^2+\dot{\mathbf{R}}\cdot\sum_i m_i\dot{\boldsymbol{\rho}}_i+\frac{1}{2}\sum_i m_i\dot{\boldsymbol{\rho}}_i^2 \\ &= \frac{1}{2}M\dot{\mathbf{R}}^2+\frac{1}{2}\sum_i m_i\dot{\boldsymbol{\rho}}_i^2 \end{aligned}$$

[1 mark per line] where the final step uses the centre of mass condition and its temporal derivative. *[3]*

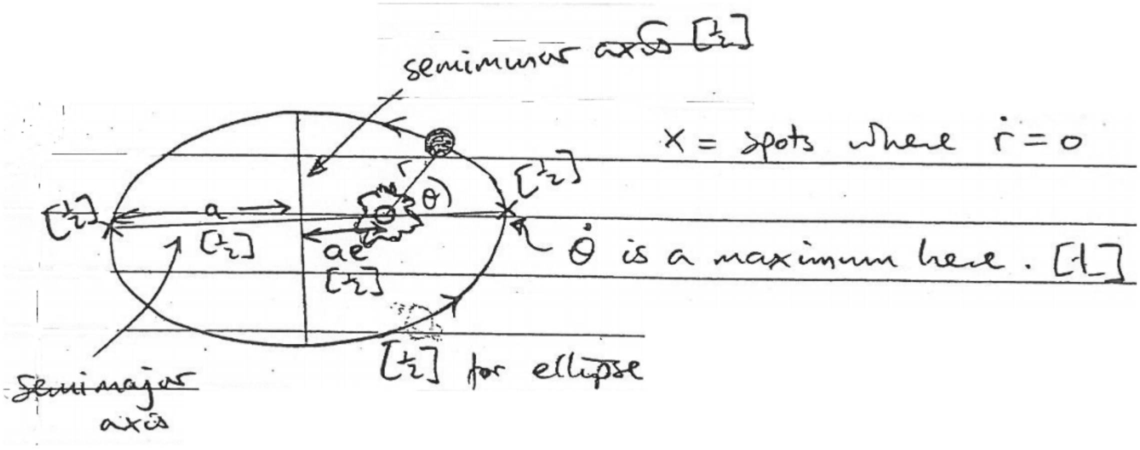
A3. Consider a lone planet in an orbit, with non-zero eccentricity e, about a remote star. Sketch the orbit, labelling

(a) the semimajor and semiminor axes, [1]

(b) the position of the star, in terms of e, along the semimajor axis, [1]

(c) the points where the planet's radial velocity momentarily vanishes, and [1]

(d) the point where the planet's angular velocity is a maximum. [1]



A4. An evil biologist surmises that if a colony of ants is kept on a rotating turntable, the insects will develop and evolve so that their left legs are stronger than their right legs. Explain the physical rationale that the mad scientist might have for this phenomenon, and the direction in which her turntable rotates. [4]

Ants on a rotating turntable will experience a Coriolis force as they travel the shortest distance relative to the turntable, in what is visually a straight line [1]. They will hence feel as if gravity were directed sideways, requiring more weight to be supported by the legs on that side [1]. *[2]*

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The Coriolis acceleration is given by $-2\omega \times \mathbf{v}$ [1]. For this to be to the left as the ant moves forwards, ω must be vertically downwards, so the turntable must rotate in a clockwise direction when viewed from above [1]. *[2]*

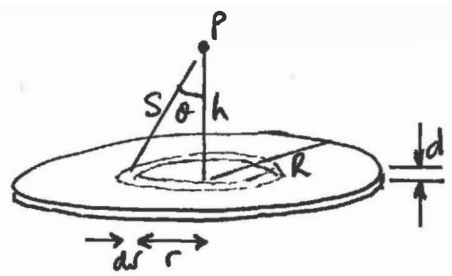
A5. Show that the gravitational field due to a horizontal uniform thin disc of thickness d, radius R and density ρ, at a distance h vertically above the disc's centre, has a magnitude

$$2\pi G \rho d \left(1 - \frac{h}{\sqrt{R^2 + h^2}}\right),$$

where G is the gravitational constant. [4]

Consider a concentric ring of radius r and cross-section $d \times \mathrm{d}r$, as shown below, whose mass will be

$$\mathrm{d}m = 2\pi r d \rho \, \mathrm{d}r.$$ *[0.5]*



The distance of all points in the ring from the field measurement point P will be $s = \sqrt{h^2 + r^2}$. The gravitational forces from each similar element of the ring will have the same vertical component but horizontal components that cancel overall, so that the total will be vertically downward. *[0.5]*

The downward component of the force upon a test mass M will have a magnitude

$$\mathrm{d}F = \frac{G\,M\,\mathrm{d}m}{s^2} \cos\vartheta = \frac{G\,M\,2\pi\,d\,\rho\,r\,\mathrm{d}r}{h^2 + r^2}\frac{h}{s} = G\,M\,2\pi\,d\,\rho\,h\,\frac{r\,\mathrm{d}r}{\left(h^2 + r^2\right)^{\frac{3}{2}}}.$$ *[1]*

The total gravitational force exerted by the disc will hence be [0.5 per line] *[1.5]*

$$\begin{aligned} F = \int_{r=0}^{r=R} \mathrm{d}F &= G\,M\,2\pi\,d\,\rho\,h \int_0^R \frac{r}{\left(h^2 + r^2\right)^{\frac{3}{2}}} \mathrm{d}r \\ &= G\,M\,2\pi\,d\,\rho\,h \left[-\left(h^2 + r^2\right)^{-\frac{1}{2}}\right]_0^R \\ &= G\,M\,2\pi\,d\,\rho\,h \left(\frac{1}{h} - \frac{1}{\sqrt{h^2 + R^2}}\right) = G\,M\,2\pi\,d\,\rho \left(1 - \frac{h}{\sqrt{h^2 + R^2}}\right). \end{aligned}$$

The gravitational field $G_D = F/M$ will hence be

$$g_D = G\,2\pi\,d\,\rho \left(1 - \frac{h}{\sqrt{h^2 + R^2}}\right).$$ *[0.5]*

Section B

B1. (a) State the vector relationship between torque and angular momentum, and explain what happens when

(i) the applied torque vector is parallel to the angular momentum, and

(ii) the applied torque is at an angle to the angular momentum. [4]

The torque $\boldsymbol{\tau}$ applied to a system is equal to the rate of change of its angular momentum $\mathbf{L}$,

$$\boldsymbol{\tau} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}.$$ *[2]*

(i) When the applied torque is parallel to the angular momentum, it simply results in an acceleration or (antiparallel) deceleration of the rotation about the angular momentum axis. *[1]*

(ii) When the torque is applied at an angle, it causes the rotation axis to change - the phenomenon of precession. *[1]*

(b) Explain what is meant by an object's *moment of inertia* about an axis, and define it mathematically in terms of the distribution of the object's mass. [2]

The moment of inertia I is the constant of proportionality between the object's angular momentum L and its angular velocity $\dot{\vartheta}$ about that axis,

$$L = I\dot{\vartheta}$$

and hence represents the reluctance of the object to change its rate of rotation in response to an applied torque τ (that is, using the result from (a), $\tau = I\ddot{\vartheta}$). *[1]*

The moment of inertia I is defined as

$$I = \int_{\text{object}} r_{\perp}^2 \,\mathrm{d}m = \int_{\text{object}} \rho(\mathbf{r})\, r_{\perp}^2 \,\mathrm{d}V$$

where $\mathrm{d}m$ is an element of mass lying a distance $r_{\perp}$ from the axis of rotation, $\rho(\mathbf{r})$ is the position-dependent density and $\mathrm{d}V$ an element of volume. [Either expression will suffice.] *[1]*

(c) An aeroplane's propeller, shown above, has a diameter of 1.88 m and mass of 14.7 kg. Making clear any assumptions, estimate

(i) its moment of inertia about the axis of the central hole; and [4]

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(ii) the torque required to increase its rate of rotation from 1200 to 2400 rpm (revolutions per minute) in 2 seconds. [2]

Assume that the mass is distributed uniformly along the propeller's length l with linear density σ. The total mass will be [1]

$$M = \int_{-D/2}^{D/2} \sigma \,\mathrm{d}l = D\sigma \qquad [\,1\,]$$

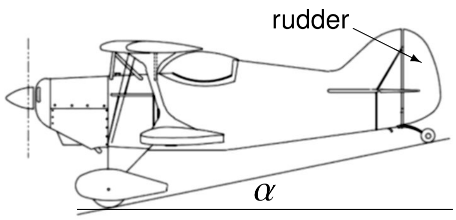
where D is the propeller's diameter. The moment of inertia will then be

$$I = \int_{-D/2}^{D/2} l^2 \sigma \,\mathrm{d}l = \frac{2}{3}\left(\frac{D}{2}\right)^3 \sigma = \frac{MD^2}{12} = 4.3\ \mathrm{kg\ m}^2. \qquad [\,2\,]$$

Assuming constant acceleration and negligible aerodynamic drag, we find [1]

$$\tau = I\ddot{\vartheta} = I\,\frac{2\pi\,(2400\,\mathrm{rpm} - 1200\,\mathrm{rpm})/(60\,\mathrm{min/sec})}{2\,\mathrm{s}} \approx 63\,I \approx 270\ \mathrm{N\ m}. \qquad [\,1\,]$$

(d) During the take-off run, the tail of the aeroplane rises, so that the aeroplane rotates through an angle $\alpha = 12^\circ$ to the attitude shown below.



Explain and calculate the effect of changing the propeller's vector angular momentum. Assume that, as viewed by the pilot, the propeller rotates anticlockwise at 3000 rpm, and that the change of attitude takes 3 s. [4]

The anticlockwise rotation of the propeller means that its angular momentum vector points aft along the aircraft's longitudinal axis. Raising the tail rotates the angular momentum vector about a lateral axis, and the change in angular momentum is roughly vertically upwards [1]. The changing angular momentum requires an upward torque, which could be produced by a rightward force at the tail, or a compensating change in the angular momentum of the rest of the aircraft, which would tend to yaw the aircraft to the right about its vertical axis [1]. [2]

The axis $\mathbf{\Omega}$ about which the aircraft rotates at a rate Ω points horizontally to the left, and is related to the torque by

$$\boldsymbol{\tau} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{\Omega} \times \mathbf{L} = \Omega\, L\,\hat{\mathbf{z}} = \Omega\, I\omega\,\hat{\mathbf{z}}$$

where ω is the angular velocity of the propeller and $\hat{\mathbf{z}}$ is an upward unit vector. Setting $\Omega = \alpha/(3\ \mathrm{s})$, we obtain

$$\boldsymbol{\tau} = \frac{12(\pi/180)}{3}\, 4.3\,\mathrm{kg\,m^2}\, \frac{2\pi\, 3000}{60} \approx 94\,\mathrm{N\,m}.$$ *[2]*

(e) To counteract this effect, the rudder deflects the airflow sideways. By considering momentum conservation, and taking the density of air to be 1.2 kg m^{-3}, estimate the cross-sectional area that the rudder must present to the airflow if it is 4.2 m from the propeller and the airspeed is 25 m s^{-1}. [4]

If the rudder presents an effective cross-sectional area A to the airflow, and maintains the speed v of the air as it is deflected relative to the aircraft through 90°, it will each second deflect a mass $\rho A v$ of air of density ρ, and thus generate a reaction force $\rho A v^2$. (This result is twice the value of the dynamic pressure $\rho v^2/2$ be derived from energy conservation.) *[2]*

For the moment of this force at a distance x to counteract the gyroscopic torque τ, we hence require the rudder to present an effective cross-sectional area

$$A = \frac{\tau}{\rho v^2 x} = \frac{94\,\mathrm{N\,m}}{1.2\,\mathrm{kg\,m^{-3}}\,(25\,\mathrm{m\,s^{-1}})^2\, 4.2\,\mathrm{m}} = 0.03\,\mathrm{m^2}.$$ *[2]*

(i.e. the 1 m-high rudder must protrude about 3 cm into the airflow. A larger deflection is needed in practice as the dynamic pressure is a factor of 2 smaller, and the airflow is not deflected by 90°.)

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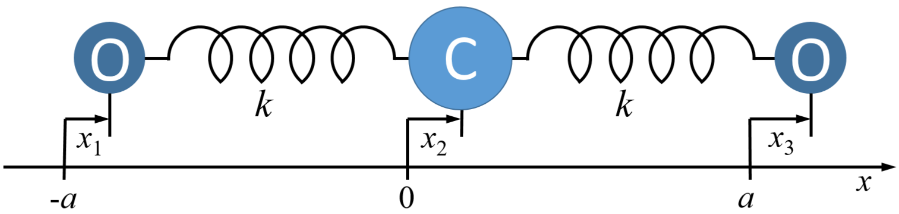
B2. (a) Explain what is meant by (i) *simple harmonic motion* and (ii) the *normal mode* of an oscillating system. [4]

(i) Simple harmonic motion is that of a single body when subject to a restoring force that is proportional to its displacement, so that the displacement varies sinusoidally with time . *[2]*

(ii) A normal mode is a motion in which all parts of the system oscillate with the same single frequency and (therefore) with a fixed phase relationship between each other. *[2]*

The stretch modes of the CO_2 molecule may be modelled by the classical system depicted below, in which the C and O atoms are rigid bobs with masses M_C and M_O, connected by two springs with spring constants k and constrained to move in only the x-direction. At rest, the O atoms are each separated from the C atom by a distance a, and x_1, x_2 and x_3 represent the displacements of the atoms from these rest positions.



(b) Setting out your working formally, derive the three equations of motion

$$
\begin{aligned}
M_O \ddot{x}_1 &= k(x_2 - x_1) \\
M_C \ddot{x}_2 &= k(x_3 - 2x_2 + x_1) \\
M_O \ddot{x}_3 &= k(x_2 - x_3)
\end{aligned}
$$

where $\ddot{x}_1 \equiv \mathrm{d}^2 x_1/\mathrm{d}t^2$, etc. [4]

The extensions of the left and right springs will be $(x_2 - x_1)$ and $(x_3 - x_2)$ respectively. *[1]*

From the definition of the spring constant k, the tensions in the two springs will hence be *[1]*

$$
\begin{aligned}
T_L &= k(x_2 - x_1) \\
T_R &= k(x_3 - x_2)
\end{aligned}
$$

so the net forces acting upon the three atoms will be *[1]*

$$
\begin{aligned}
F_1 &= T_L \\
F_2 &= T_R - T_L \\
F_3 &= -T_R.
\end{aligned}
$$

Applying Newton's second law for each of the atoms, we hence obtain the equations of motion *[1]*

$$\begin{aligned} M_O\ddot{x}_1 &= F_1 = k(x_2 - x_1) \\ M_C\ddot{x}_2 &= F_2 = k(x_3 - 2x_2 + x_1) \\ M_O\ddot{x}_3 &= F_3 = k(x_2 - x_3). \end{aligned}$$

(c) By substituting the normal mode solutions $x_j = a_j \exp(\mathrm{i}\omega t)$, where $j = 1 \ldots 3$, show that the common frequency of motion ω must satisfy

$$\omega^2 \left(M_O\omega^2 - k\right) \left[M_C M_O\omega^2 - (M_C{+}2M_O)k\right] = 0.$$ [8]

Substituting the given expressions $x_i = a_i \exp(\mathrm{i}\omega t)$ into the equations of motion, and cancelling the common factor $\exp(\mathrm{i}\omega t)$, we obtain *[2]*

$$\begin{aligned} -\omega^2 M_O a_1 &= k(a_2 - a_1) \\ -\omega^2 M_C a_2 &= k(a_3 - 2a_2 + a_1) \\ -\omega^2 M_O a_3 &= k(a_2 - a_3) \end{aligned}$$

which may be rearranged and written in matrix form

$$\begin{pmatrix} M_O\omega^2 - k & k & 0 \\ k & M_C\omega^2 - 2k & k \\ 0 & k & M_O\omega^2 - k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0.$$ *[1]*

This requires (since a_{1-3} cannot be obtained by operating upon the zero vector, the matrix must be non-invertible) that

$$\begin{vmatrix} M_O\omega^2 - k & k & 0 \\ k & M_C\omega^2 - 2k & k \\ 0 & k & M_O\omega^2 - k \end{vmatrix} = 0$$ *[1]*

which yields, from calculation of the determinant,

$$(M_O\omega^2 - k)\left[(M_C\omega^2 - 2k)(M_O\omega^2 - k) - k^2\right] - k\left[k(M_O\omega^2 - k)\right] = 0.$$ *[1]*

Collecting the factors of $(M_O\omega^2 - k)$ hence gives

$$(M_O\omega^2 - k)\left[(M_C\omega^2 - 2k)(M_O\omega^2 - k) - 2k^2\right] = 0$$ *[1]*

which, upon expansion of the product and cancellation of terms in k^2, gives

$$(M_O\omega^2 - k)\left[M_C M_O\omega^4 - k\omega^2(M_C + 2M_O)\right] = 0.$$ *[1]*

This can now be arranged as the product of factors given,

$$\omega^2(M_O\omega^2 - k)\left[M_C M_O\omega^2 - (M_C + 2M_O)k\right] = 0.$$ *[1]*

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(d) Hence find expressions for the frequencies $\omega_{\rm asym}$ and $\omega_{\rm sym}$ of the *asymmetric stretch* and *symmetric stretch* modes, and comment upon how their ratio compares with the measured value of $\omega_{\rm asym}/\omega_{\rm sym} = 1.69$. [4]

The roots to the above equation are $\omega^2 = 0$, $\omega^2 = k/M_O$ and $\omega^2 = [(M_C+2M_0)/M_C]k/M_O$. These correspond respectively to the common mode (steady motion), symmetric stretch and asymmetric stretch - that is, *[2]*

$$\begin{aligned}\omega_{\rm sym} &= \sqrt{\frac{k}{M_O}}\\ \omega_{\rm asym} &= \sqrt{\frac{M_C+2M_0}{M_C}}\sqrt{\frac{k}{M_O}}.\end{aligned}$$

The ratio of the frequencies of the asymmetric and symmetric stretch modes is hence

$$\frac{\omega_{\rm asym}}{\omega_{\rm sym}} = \sqrt{\frac{M_C+2M_0}{M_C}} = \sqrt{1+2\frac{M_0}{M_C}} = \sqrt{1+2\times\frac{4}{3}} = 1.92.$$ *[1]*

The calculated ratio is inconsistent with the experimental value, which proves to be complicated by bending motion not permitted in the linear model. *[1]*

The ratio of the atomic masses, M_O/M_C, is approximately $4/3$.

B3. (a) Explain what are meant by *centrifugal* and *Coriolis* forces. Outline their origins, the properties upon which they depend, and the directions in which they act. [4]

The centrifugal and Coriolis forces are not associated with attraction or repulsion between bodies, but are apparent in non-inertial reference frames [0.5] as a result of the rotational variation of the coordinate axes [0.5].

The centrifugal force depends only upon the position in and angular velocity of the rotating coordinate frame [0.5], and acts radially outward from the axis of rotation [0.5], accounting for the need (in an inertial frame) of a centripetal force to maintain the radial coordinate [0.5].

The Coriolis force is the additional effect of motion within the rotating frame and a consequence of the conservation of angular momentum [0.5]; it depends upon, and acts in a direction normal to, both the local velocity and the angular velocity of the rotating frame [1].

The crew cabin aboard the spaceship *Discovery One* occupies the inner rim of a cylinder 16 m in diameter, which rotates steadily to provide an apparent gravity close to that on Earth. A running track between the workstations and rest areas allows astronauts to exercise.

(b) Calculate the angular velocity with which the cylinder should rotate. [2]

For the effective gravity to match the Earth's value of $g = 9.8\,\mathrm{m\,s^{-2}}$ at a radius a, the cylinder's angular velocity ω must satisfy

$$a\,\omega^2 = g. \qquad [\,1\,]$$

With the values given, this requires $\omega = \sqrt{g/a} = \sqrt{9.8\mathrm{m\,s^{-2}}/(16\,\mathrm{m}/2)} = 1.1\,\mathrm{rad\,s^{-1}} \equiv 10.6\,\mathrm{rpm}.$ *[1]*

(c) The crew members quickly realize that it is easier to run in one direction than the other. Explain this observation, and estimate the difference quantitatively for a running speed of 2.5 m s^{-1}. [3]

The effective gravity depends upon the runner's rotational speed, which is raised or lowered by his/her motion relative to the rotating cylinder. *[1]*

At a radius of 8 m, running at 2.5 m s^{-1} changes the runner's angular velocity by 2.5 m s$^{-1}/8$ = $0.3\,\mathrm{rad\,s^{-1}}$ *– about a third of the angular velocity of the cylinder. If the runner runs against the rotation of the cylinder, his/her apparent gravity will be reduced to* $8 \times (1.1 - 0.3)^2 = 5\,\mathrm{m\,s^{-2}}$, *whereas if he/she runs in the direction of rotation it will be increased to* $8 \times (1.1+0.3)^2 = 16\,\mathrm{m\,s^{-2}}$. *[2]*

(d) The air within the rim of the cylinder is maintained at approximately the Earth's atmospheric density of $1.2\,\mathrm{kg\,m^{-3}}$. Assuming that the air is uniform in temperature and density, and rotates with the cylinder, find the pressure difference between the edge of the cylinder and its axis. [5]

Let $P(r)$ *be the pressure at a radius* r *from the cylinder's axis, where the apparent gravity will be* $g(r) = r\,\omega^2$. *[1]*

Across an elemental difference in radius δr*, the difference in pressure acting over elemental areas* $(r\,\delta\theta) \times \delta z$ *must support the weight of the air enclosed. Hence, if the air density is* ρ*,*

$$[P(r+\delta r) - P(r)]\,r\,\delta\theta\,\delta z = \rho\,\delta r\,r\,\delta\theta\,\delta z\,(r\,\omega^2). \qquad [\,1\,]$$

Dividing by $r\,\delta\theta\,\delta z\,\delta r$ *and taking the limit as* $\delta r \to 0$*, we obtain*

$$\frac{\mathrm{d}P}{\mathrm{d}r} = \lim_{\delta r \to 0} \frac{[P(r+\delta r) - P(r)]}{\delta r} = \rho\,r\,\omega^2. \qquad [\,1\,]$$

Integration from $r = a$ *now yields*

$$P(r) - P(a) = \rho\,\omega^2\,\frac{r^2 - a^2}{2}. \qquad [\,1\,]$$

With $(a - r) = 8\,\mathrm{m}$*,* $\omega = 1.1\,\mathrm{rad\,s^{-1}}$ *and* $\rho(a) = 1.2\,\mathrm{kg\,m^{-3}}$*, we obtain*

$$P(a) - P(0) = 8^2\,\frac{(1.1\,\mathrm{rad\,s^{-1}})^2\,(1.2\,\mathrm{kg\,m^{-3}})}{2} = 47\,\mathrm{Pa}. \qquad [\,1\,]$$

(e) The cylinder and associated equipment have a total mass of 50 000 kg. Find the magnitude and direction of the gravitational acceleration that would be experienced at the rim if this mass were concentrated at the cylinder's axle. [3]

The gravitational acceleration δg *due to this mass* M *would be*

$$\delta g = \frac{GM}{a^2} = \frac{(6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}})\,(50\,000\,\mathrm{kg})}{(8\,\mathrm{m})^2} = 5 \times 10^{-8}\,\mathrm{m\,s^{-2}}. \qquad [\,2\,]$$

This acts towards the position of the mass – i.e., towards the centre of the cylinder. *[1]*

(f) Explain how your answer to (e) would differ qualitatively if the mass were instead uniformly distributed around the cylindrical rim? [3]

The gravitational field would be outwards. *[1]*

Unlike a spherical distribution, a thin cylindrical mass distribution has a non-zero internal field, for the effect of the nearby mass is incompletely cancelled by the more distant mass opposite: even for a thin cylinder, one cannot construct a closed 'Gaussian surface' that preserves the cylindrical symmetry and is divisible into areas of uniform field. *[1]*

One approach would be to complete the cylinder into a (roughly) spherical shell by the addition of mass at smaller radii than the observer. Since the total field within the spherical shell must be zero, the field of the cylinder alone must equal and oppose that of the added mass, which by symmetry must be towards the cylinder's centre. *[1]*

[The gravitational attraction of a thin cylindrical mass increases monotonically as the rim is approached, until the distance is comparable with the width and thickness of the rim.]

The gravitational constant $G = 6.67 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$.

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B4. A spacecraft follows an elliptical orbit about the Earth and, while its rocket engines are inactive, is subject only to the Earth's gravitational attraction.

(a) Explain why the vector angular momentum is conserved and why this means that the orbit lies in a plane. [4]

The gravitational attraction is a central force, so it exerts no torque about the Earth on the satellite [1]. Since the torque defines the rate of change of angular momentum, the angular momentum of the satellite about the Earth is constant [1]. *[2]*

Since the angular momentum $\mathbf{L}$ *is conserved, and always perpendicular to the position* $\mathbf{r}$ *and its rate of change* $\dot{\mathbf{r}} = \mathbf{p}/m$ *[1], the satellite's motion remains in the plane defined by the initial* $\mathbf{r}$ *and* $\mathbf{p}$ *[1].* *[2]*

(b) What other quantity or quantities is/are conserved, and why? [2]

Since the gravitational attraction is a central, spherically symmetric force [1], the total energy [1] of the satellite will be conserved. *[2]*

The spacecraft's orbit is initially in the equatorial plane of the Earth, but must be changed to an elliptical orbit of the same eccentricity in a polar plane (i.e. from a plane normal to the Earth's rotation axis to one containing it).

(c) At which point in the elliptical orbit can the change of orbit be accomplished most efficiently? Explain your answer. [3]

The change of orbit requires a change in angular momentum [1]. This is accomplished most efficiently at the apogee [1] (furthest from the Earth), as the thrust of the spacecraft's rocket motor can there exert the greatest moment or torque [1]. *[3]*

(d) The change of orbit is accomplished by firing the spacecraft's rocket motor for a time that is much less than the orbital period. How will the spacecraft's velocities $\mathbf{v}_i$ and $\mathbf{v}_f$ immediately before and after the impulse be related? [2]

The spacecraft will be at the same point within an elliptical orbit of the same shape, so its speeds before and after will be the same, i.e. $|\mathbf{v}_i| = \left|\mathbf{v}_f\right|$. *[1]*

As the orbital planes are orthogonal, however, the velocities will be perpendicular, i.e. $\mathbf{v}_i \cdot \mathbf{v}_f = 0$. *[1]*

(e) Show that, if at time t the spacecraft has a total mass $m(t)$ and ejects exhaust gas from its rocket motor with a relative velocity $\mathbf{u}$, then its velocity $\mathbf{v}(t)$ satisfies

$$m\,\mathrm{d}\mathbf{v} = -\mathbf{u}\,\mathrm{d}m. \qquad [\,2\,]$$

Equating the total momenta of the spacecraft and exhaust before and after ejection of an infinitessimal mass $\mathrm{d}m$ that results in a velocity increase $\mathrm{d}\mathbf{v}$,

$$m\mathbf{v} = (m - \mathrm{d}m)\,(\mathbf{v} + \mathrm{d}\mathbf{v}) + \mathrm{d}m\,(\mathbf{v} + \mathbf{u})\,. \qquad [\,1\,]$$

Expanding this expression, cancelling terms, and neglecting the term $\mathrm{d}m\mathrm{d}\mathbf{v}$, which will be of vanishing significance for infinitessimal changes, we obtain

$$m\,\mathrm{d}\mathbf{v} = -\mathbf{u}\,\mathrm{d}m. \qquad [\,1\,]$$

(f) Hence, noting any assumptions, show that the initial and final velocities $\mathbf{v}_i$ and $\mathbf{v}_f$ are related to the initial and final masses m_i and m_f by

$$\mathbf{v}_f = \mathbf{v}_i + \mathbf{u}\ln\frac{m_i}{m_f}. \qquad [\,2\,]$$

Assuming $\mathbf{u}$ to be constant, this expression may be rearranged to give

$$\mathrm{d}\mathbf{v} = -\mathbf{u}\frac{\mathrm{d}m}{m} \qquad [\,1\,]$$

which can be integrated to give

$$\mathbf{v}_f - \mathbf{v}_i = -\mathbf{u}\left(\ln m_f - \ln m_i\right)$$

hence

$$\mathbf{v}_f = \mathbf{v}_i + \mathbf{u}\ln\frac{m_i}{m_f}. \qquad [\,1\,]$$

(g) The spacecraft has an empty mass of 950 kg and is initially in a nearly circular orbit of radius 7230 km. Find the minimum mass of fuel that must be burned if the exhaust gas leaves with a relative speed of 3120 m s^{-1}. [5]

In a circular orbit of radius r about the Earth of mass M,

$$\frac{m\,v^2}{r} = \frac{G\,m\,M}{r^2}. \qquad [\,1\,]$$

TURN OVER

The orbital speed will hence be

$$v = \sqrt{\frac{G\,M}{r}}. \qquad [\,1\,]$$

The required change in velocity will be

$$\left|\mathbf{v}_f - \mathbf{v}_i\right| = \sqrt{2}v = \sqrt{\frac{2\,G\,M}{r}}. \qquad [\,1\,]$$

Rearranging the solution to (f), we obtain

$$m_i - m_f = m_f\left[\exp\frac{\left|\mathbf{v}_f - \mathbf{v}_i\right|}{u} - 1\right] = m_f\left[\exp\frac{\sqrt{\frac{2\,G\,M}{r}}}{u} - 1\right] \qquad [\,1\,]$$

so, with the values given and assuming the fuel to be exhausted after the manoeuvre, we find the required fuel mass to be

$$m_i - m_f = 950\,\mathrm{kg}\left[\exp\frac{\sqrt{\frac{2(6.67\times10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}})\,(5.97\times10^{24}\,\mathrm{kg})}{7.27\times10^{6}\,\mathrm{m}}}}{3120\,\mathrm{m\,s^{-1}}} - 1\right] = 26\,300\,\mathrm{kg}. \qquad [\,1\,]$$

[This is rather more than the 6000 kg fuel load of the Delta K second-stage spacecraft used to deliver the NOAA-19 polar-orbiting meteorological satellite which inspired this question: presumably the change in orbital plane begins in the first stage of the delivery trajectory.]

The mass of the Earth may be taken to be 5.97×10^{24} kg, and the gravitational constant to be $G = 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$.

END OF PAPER