SEMESTER 1 EXAMINATION 2017-2018

WAVE PHYSICS

Duration: 120 MINS (2 hours)

This paper contains 9 questions.

Answers to Section A and Section B must be in separate answer books

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language dictionary is permitted ONLY IF it is a paper version of a direct ‘Word to Word’ translation dictionary AND it contains no notes, additions or annotations.

Section A

A1. Explain what are meant by the terms *travelling* and *standing* waves. [2]

Show, with a mathematical example, how travelling waves may be superposed to form a standing wave, and vice-versa. [2]

A2. What is meant by wave *interference*? [1]

Describe how interference occurs in a *Michelson interferometer*, and outline how and for what purpose such an instrument may be used. Illustrate your answer with a simple diagram. [3]

A3. Explain what is meant by the *impedance* of a medium in the context of wave propagation. [2]

A wave is incident at right-angles upon the interface between two media. Show, with the aid of a formula, how the fraction of the wave power that is reflected depends upon the impedances of the two media. [2]

A4. The energy density of a sound wave is, like other mechanical wave motions, composed in part by the kinetic energy $\frac{1}{2}\rho(\partial\xi/\partial t)^2$ and in part by the potential energy $\frac{1}{2}E(\partial\xi/\partial x)^2$, where ξ is the longitudinal displacement at position x and time t, ρ is the density of the medium and E its modulus of elasticity.

(a) By considering a sinusoidal sound wave of definite frequency, show that these two contributions are equal. [2]

(b) Given that the acoustic intensity is equal to the product of mean energy density and wave speed, find the amplitude of displacement for sound waves in sea water that correspond to the limit of dolphin hearing, $10^{-14}\,\mathrm{W\,m^{-2}}$ at a frequency of 50 kHz. The density of sea water may be taken to be 1025 kg m^{-3}, the modulus of elasticity $E \approx 2.3 \times 10^9$ Pa, and the wave speed is given by $v = \sqrt{E/\rho}$. [2]

For the sound waves addressed here, the phase velocity $v_p = \sqrt{E/\rho}$.

A5. What is meant by an *operator* in the context of wave motion? [2]

Express the frequency and wavenumber operators $\hat{\omega}$ and $\hat{k}$ in terms of differential functions and, for one of them, show that application to a complex exponential travelling wave correctly yields the wave frequency and wavenumber. [2]

Section B

B1. (a) Making clear any assumptions, derive the equations governing the flow of heat down a uniform, thin metal bar,

$$\begin{aligned} Q(x) &= -\kappa A \frac{\partial \Theta}{\partial x} \\ \text{and} \qquad \frac{\partial \Theta}{\partial t} &= -\frac{1}{C\rho A}\frac{\partial Q}{\partial x}, \end{aligned}$$

where A is the cross-sectional area of the bar, x is the coordinate measured along its length, $\Theta(x)$ and $Q(x)$ are respectively the temperature and rate of heat flow along the bar, C is the specific heat capacity of the metal, ρ its density and κ its thermal conductivity. [6]

(b) Hence derive the *diffusion* wave equation

$$\frac{\partial \Theta}{\partial t} = \frac{\kappa}{C\rho}\frac{\partial^2 \Theta}{\partial x^2}.$$ [1]

(c) Show that the diffusion wave equation does not have sinusoidal wave solutions, but may be solved by complex exponential waves of the form

$$\Theta(x,t) = \Theta_0 \exp\left[\mathrm{i}(kx - \omega t)\right]$$

and derive the *dispersion relation*

$$k(\omega) = \pm(1+\mathrm{i})\sqrt{\frac{C\rho}{2\kappa}}\ \sqrt{\omega}.$$ [5]

(d) Hence derive the real, forward-travelling solution of the form

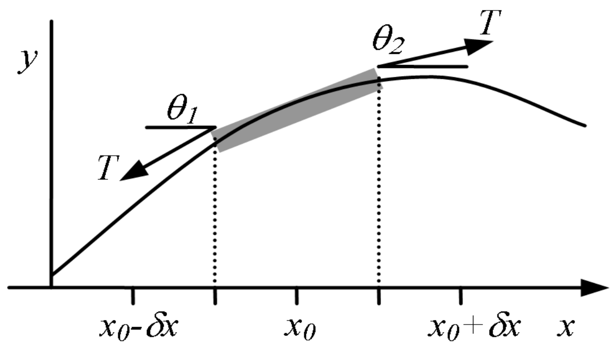
$$\Theta(x,t) = \Theta_0 \cos(\pm k_0 x - \omega t + \varphi)\exp(\mp k_0 x),$$

and show that $k_0 = \sqrt{C\rho\omega/(2\kappa)}$. [4]

(e) The end of a brass bar is heated and cooled so that its temperature alternates with 1-second period between two fixed values. Describe how the temperature variations will differ at different distances along the bar, and, given that for brass $C\rho/\kappa = 29\,180\,\mathrm{s\,m^{-2}}$, find the speed with which points of maximum temperature eventually propagate. [4]

B2. (a) What is meant by the *Fourier transform*? How may it be defined? [4]

The figure below shows a section of a thin, flexible string of mass per unit length ρ and subject to a tension T.



(b) By considering the net force acting on an element of the string (which, shown in grey, may be considered approximately rigid), derive the wave equation governing its transverse motion,

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}.$$ [5]

(c) By substituting sinusoidal travelling waveform $y(x, t) = y_0 \cos(kx - \omega t + \varphi)$ into the wave equation, find the dispersion relation between k and ω, and hence show that the speed of propagation along the taut string is $\sqrt{T/\rho}$. [3]

(d) Hence find the tension of a guitar string of length 520 mm and linear density 0.002 kg m^{-1} when tuned to a frequency of 220Hz (the note A_3). [2]

Ed Sheeran uses his loop station to record a rhythm track that comprises three bursts of the same guitar note A_3. The notes each last 1 s and are separated by 0.25 s, with a pause of 1.5 s before the sequence ends. The resulting 5 s track is repeated continuously throughout his song. A recording engineer uses a high resolution spectrum analyser to monitor the intensity spectrum of the loop track.

(e) Sketch the recorded signal as a function of time. [2]

(f) Hence sketch, in as much detail as possible, the spectrum that the engineer would obtain. [4]

TURN OVER

B3. (a) Explain what is meant by the *Doppler effect*. [2]

A source of waves of angular frequency ω_s moves with a velocity $\mathbf{v}$ and, at time $t = 0$, is at a position $\mathbf{r}_0$ relative to a stationary observer.

(b) Show, with the aid of a diagram, that the distance from the source to the observer at time $t \ll |\mathbf{r}_0|/|\mathbf{v}|$ will be given approximately by

$$r \approx |\mathbf{r}_0| + \frac{\mathbf{r}_0}{|\mathbf{r}_0|} \cdot \mathbf{v}t.$$ [4]

(c) Show therefore that if the wave leaving the source at time t is $\psi(t)$, then that seen by the observer will be proportional to

$$\psi\left(t - t_0 - \frac{\mathbf{r}_0}{|\mathbf{r}_0|} \cdot \frac{\mathbf{v}}{c} t\right),$$

where $t_0 = |\mathbf{r}_0|/c$ and c is the speed with which the wave propagates. [3]

(d) Hence show that the observed wave will have an angular frequency $\omega_s + \delta\omega$, where

$$\frac{\delta\omega}{\omega_s} = -\frac{v_x}{c},$$

and v_x is the component of the source's velocity away from the observer. [4]

The source is an atom which, when at rest, emits or scatters photons of angular frequency ω_0. The atom emits a photon towards the observer, in whose frame the photon has an energy $\hbar\omega$. The coordinate axes may be chosen so that the x axis points from the observer to the source. Conservation of energy and momentum upon the emission of the photon leads to the relations

$$\begin{aligned} \hbar\omega &= \hbar\omega_0 - m\, v_x\, \delta v \\ m\,\delta v &= \hbar\omega/c, \end{aligned}$$

where δv is the change in the x-component of the atom's velocity when it emits the photon, m is the mass of the atom and v_x is the mean component of its velocity away from the observer.

(e) Show that the observed angular frequency will be

$$\omega = \omega_0 \left(1 + \frac{v_x}{c}\right)^{-1}.$$ [2]

(f) Show therefore that, if $v_x \ll c$, the Doppler shift of the photon due to the motion of the atom will again be

$$\delta\omega = \omega - \omega_0 \approx -\omega_0 \frac{v_x}{c}.$$ [3]

The Fraunhofer E line in the solar spectrum is due to absorption at wavelength $\lambda_0 = 527$ nm by Fe atoms in the photosphere, where the temperature T is around 5 000 K.

(g) Estimate the variation $\delta\lambda$ in the wavelength of the E line that is due to thermal motion of the Fe atoms. The r.m.s. velocity component $v_{x,rms}$ for a thermal distribution is given by $v_{x,rms}^2 = k_B T/m$, where k_B is Boltzmann's constant and the mass m of a Fe atom is 9.27$\times 10^{-26}$ kg. [2]

You may assume that $\delta\lambda/\lambda_0 = \delta\omega/\omega_0$.

TURN OVER

B4. (a) Explain what is meant by *Fraunhofer diffraction*. [3]

(b) Show from first principles, with the aid of a diagram, that the dependence upon angle of the relative amplitude of a wave of wavelength λ, diffracted by a single slit of width a is given by the sinc function

$$a_1(\vartheta) \propto \frac{\sin\left\{\frac{\pi a}{\lambda}\sin\vartheta\right\}}{\frac{\pi a}{\lambda}\sin\vartheta}.$$ [5]

(c) State the *convolution theorem* and explain how it may be used to determine the diffraction patterns of regular arrays of a basic pattern. [3]

The diffraction pattern of an infinite regular array of narrow slits, whose centres are separated by a distance d, is given by

$$a_2(\vartheta) = a_0 \sum_{n=0}^{\infty} \delta\left(\vartheta - n\frac{\lambda}{d}\right)$$

where ϑ is the angle through which the incident beam is diffracted.

(d) Write the transmission of a real diffraction grating, of width b and composed of narrow slits of width c spaced by a distance d, as a combination of products and convolutions of simple functions; [2]

(e) Hence determine and sketch the diffraction pattern of such a grating. [4]

(f) Given that for small angles ($\vartheta \sim \sin\vartheta$), adjacent diffraction orders are separated in angle by λ/d, and the width of each order is around λ/b, estimate the theoretical resolution of a grating spectrograph for use in first order at $\lambda = 2000$ nm when the grating parameters are $b = 100$ mm, $d = (1/300)$ mm, $c = 0.3\ \mu$m. [3]

END OF PAPER