WAVE PHYSICS exam January 2008.

A1. Transverse waves involve the propagation of a particle or medium displacement, or of a field component, in a direction perpendicular to the direction of wave propagation. [1 mark]

Longitudinal waves involve the propagation of a displacement or field component parallel to the direction of propagation. [1 mark]

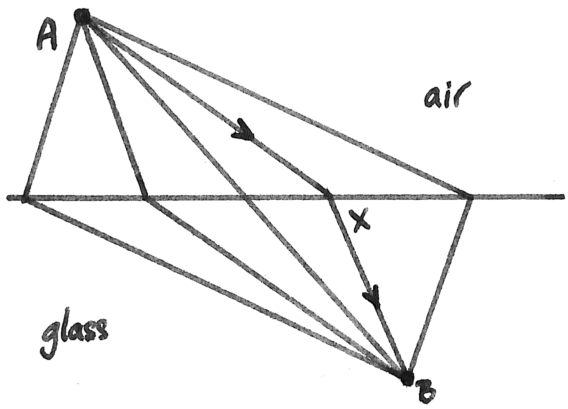
Examples: transverse waves: electromagnetic radiation
gravitational waves
guitar string
shallow water waves [$\frac{1}{2}$ mark]

longitudinal waves: sound
thermal waves [$\frac{1}{2}$ mark]

neither: quantum wavefunctions
arguably torsion, spin, coax cable
chemical/biological/neurological eg conduction
possibly Mexican waves, 'waves of fear' [1 mark]

Note that longitudinal components of e-m, gravitational and guitar waves are possible; shallow water waves involve a small longitudinal component

A2. Fermat: in travelling between two points, a wave follows the path which takes the least time. [1 mark]



[1 mark]

In comparison with the straight line route, the path via X, although longer, involves spending less time in the slow medium (glass), the saving being greater than the increase in time spent in air to get there. [1 mark]

The path followed will be AXB. [1 mark]

A3. For the greatest discomfort we're told that we need waves with

wavelength $= 2 \times$ boat length $= 2 \times 20.8\,m = 41.6\,m.$ [$\frac{1}{2}$ mark]

frequency $=$ resonant frequency $= 1/4s = 0.25\,Hz.$ [$\frac{1}{2}$ mark]

$\Rightarrow$ the wave speed for greatest discomfort will be

$$41.6\,m \times 0.25\,s^{-1} = 10.4\,m.s^{-1}$$ [1 mark]

The wave speed depends upon the water depth via $v = \sqrt{gh}$

so $$\sqrt{gh} = 10.4\,m.s^{-1}$$

ie. $$h = \frac{(10.4\,m.s^{-1})^2}{g} = \frac{10.4^2}{9.81}\,m = \underline{\underline{11.0\,m.}}$$ [1 mark]

(Note that $\lambda \sim 4h$, so we're reasonably into the 'shallow water' regime.)

If the boat is heading at an angle θ to the wavevector then, assuming no leeway, the effective boat length seen by the waves will be $L\cos\theta$, where L is the actual length, and the apparent wave velocity will become $v - v_L\cos\theta$. We hence obtain, for a resonant period τ,

$$\sqrt{gh} - v_L\cos\theta = \frac{2L\cos\theta}{\tau}$$

so $$h = \frac{(2L/\tau + v_L)^2\cos^2\theta}{g}.$$

ie. the depth increases with v_L but decreases with θ. [1 mark]

(A simple non-mathematical summary will suffice.)

A4. The principle of Fourier synthesis is that any waveform can be built up from sinusoidal wave components of appropriate magnitudes, frequencies and phases. [1 mark]

The principle of Fourier analysis is that any waveform can be broken down into such components, by exploiting the orthogonality of sine waves of different frequency. [1 mark]

In linear systems, the propagation of an arbitrary waveform may therefore be determined by determining the Fourier sinusoidal components, allowing for the behaviour of each, and recombining the results to re-form the composite wave. [1 mark] [1 mark]

A5. $y(x,t) = y(\phi)$ where $\phi = ax - bt$:

$y(x,t) = C$, constant implies ϕ = constant, ϕ_0

so $\phi = ax - bt = \phi_0$

$\Rightarrow \quad x = \frac{b}{a}t + \frac{\phi_0}{a}$ [1 mark]

$\Rightarrow \quad \frac{dx}{dt} = \underline{\underline{\frac{b}{a}}}$, the propagation speed of the wave. [1 mark]

In terms of ω and k,

$\phi = kx - \omega t$, ie. $a \equiv k$, $b \equiv \omega$

$\Rightarrow$ phase velocity $\underline{\underline{v_p = \omega/k}}$. [1 mark]

The group velocity, $v_g = d\omega/dk$, is the propagation speed of the envelope of a superposition of several frequencies of sinusoidal components, when the phase velocity varies with frequency. [1 mark]

B1. Travelling waves maintain a constant form that is simply translated through space as time proceeds. [1 mark]

Standing waves maintain a spatially fixed form, that is multiplied by an evolving function of time. [1 mark]

Standing waves may be written as equal superpositions of counter-propagating travelling waves and thus, unlike travelling waves, cannot be identified with a particular flux of energy or momentum. [1 mark]

(Travelling waves may be written as equal superpositions of standing waves that act in quadrature.)

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}$$

We try $y(x,t) = f(u)$ where $u = x - vt$, so that

$$\frac{\partial y}{\partial x} = \frac{dy}{du}\frac{\partial u}{\partial x} = \frac{dy}{du}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{d}{du}\left(\frac{dy}{du}\right)\frac{\partial u}{\partial x} = \frac{d^2 y}{du^2}$$

$$\frac{\partial y}{\partial t} = \frac{dy}{du}\frac{\partial u}{\partial t} = -v\frac{dy}{du}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{d}{du}\left(-v\frac{dy}{du}\right)\frac{\partial u}{\partial x} = v^2\frac{d^2 y}{du^2}$$

$$\Rightarrow \quad v^2\frac{d^2 y}{du^2} = \frac{T}{\rho}\frac{d^2 y}{du^2}$$

$\Rightarrow$ <u>assumed form is a solution, provided that $v^2 = T/\rho$.</u> [2 marks]

B1 cont'd. Hence $\underline{\underline{v = \pm\sqrt{T/\rho}}}$. [1 mark]

(a) At $t=0$, $y(x,0)$ has the form sketched, which we may call $y_0(x)$; [1 mark]

$$\frac{\partial y}{\partial t}(x,0) = 0$$ [1 mark]

(b) If $y(x,t) = y_+(x - v_+ t) + y_-(x - v_- t)$

$$= y_+(x - Vt) + y_-(x + Vt) \quad \text{where } V = \sqrt{T/\rho},$$

then the conditions (a) require

$$y_+(x) + y_-(x) = y_0(x) \qquad \text{(i)}$$ [1 mark]

$$-V\frac{\partial y_+}{\partial x}(x,0) + V\frac{\partial y_-}{\partial x}(x,0) = 0 \qquad \text{(ii)}$$ [1 mark]

From (ii): $$\frac{\partial y_+}{\partial x}(x,0) = \frac{\partial y_-}{\partial x}(x,0)$$

$$\Rightarrow \quad y_+(x,0) = y_-(x,0) + \text{constant}$$ [1 mark]

where the constant of integration may be set to zero without affecting the overall solution. Combining with (i), we hence obtain

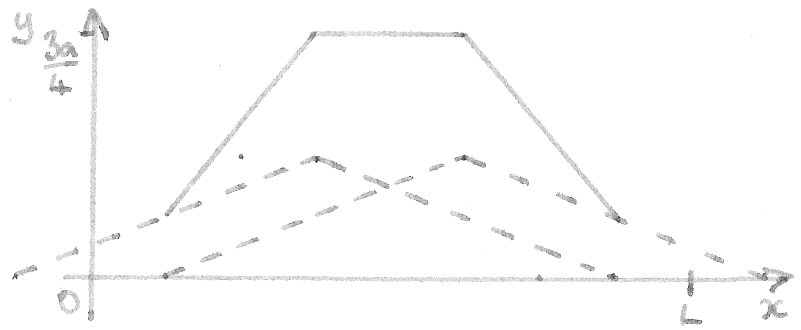
$$y_+(x,0) = y_-(x,0) = \tfrac{1}{2}y_0(x)$$

and thus

$$\underline{\underline{y_+(x - v_+ t) = y_-(x - v_- t) = \tfrac{1}{2}y_0(x).}}$$ [1 mark]

B1 cont'd. (c) At time $t = L/8v_{\pm}$ the two components will be displaced by

$\pm v_{\pm}t = \pm L/8$ from their initial positions. Hence, in the range given, they and their superposition will be as shown.



[2 marks]

Boundary conditions: $y(0,t) = y(L,t) = 0$. [1 mark]

$\Rightarrow y_+(-vt) + y_-(vt) = 0;$ (iii)

$y_+(L-vt) + y_-(L+vt) = 0$ (iv)

from (iii): $y_-(L+vt) = -y_+(-(L+vt)) = -y_+(-L-vt)$ (v)

from (iv): $y_+(L-vt) = y_+(-L-vt)$

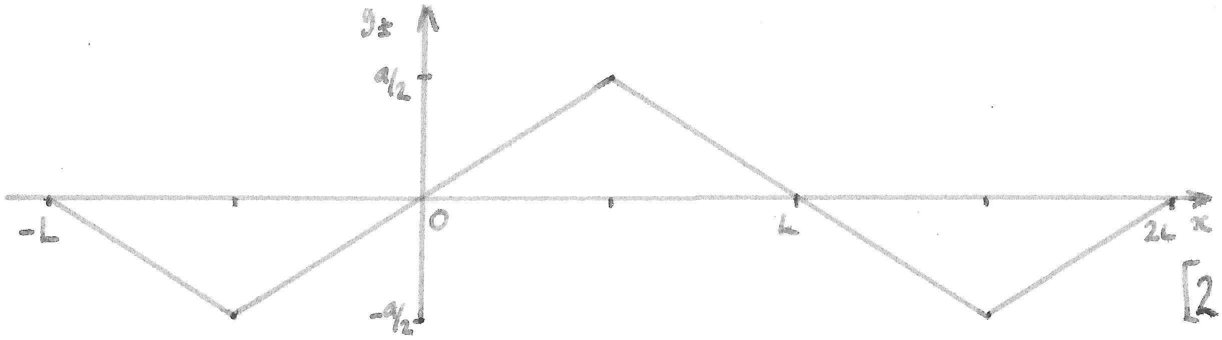
therefore the functions $y_{\pm}$ are periodic with periodicity $2L$. [1 mark]

Since $y_+(x) = y_-(x)$, (iii) gives

$y_+(x) = -y_+(-x)$

therefore the functions $y_{\pm}$ are antisymmetric about $x=0$ (or, equivalently, about $x=L$). [1 mark]

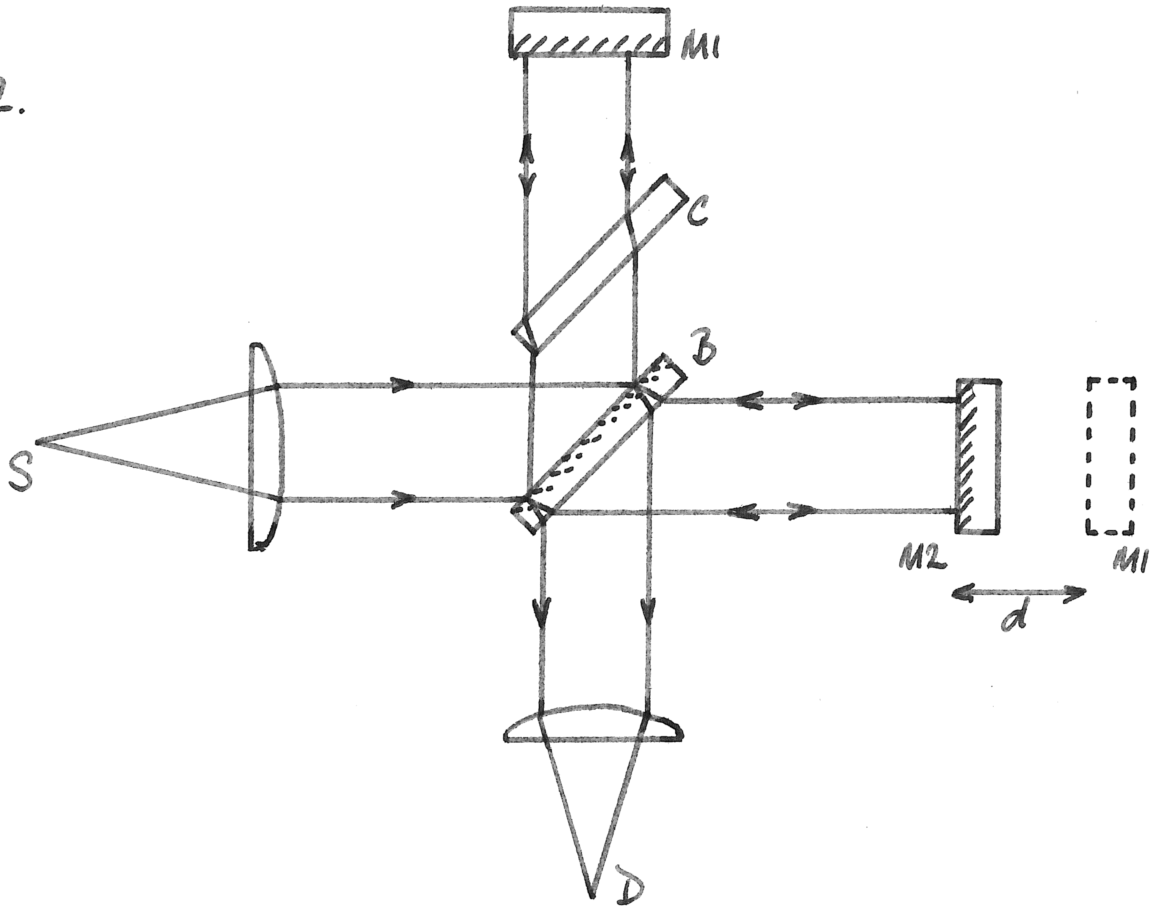
B1 cont'd. Hence $y_\pm(x,0)$ are as below:



[2 marks]

The sketch for $t = L/8v_+$ is completed by extending the superposition linearly to intersect $(0,0)$ and $(L,0)$. [1 mark]

B2.

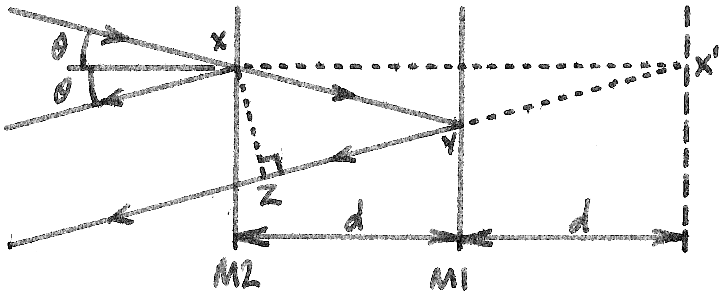


[2 marks]

The Michelson interferometer works by interfering two beams of light obtained from the same source by division of amplitude, which reach the detector by paths of different length.
Collimated light from the source S strikes the partially-reflecting beam-splitter B. Part of the light is reflected, and passes to mirror M1, reflecting it back along its path; the other part is transmitted, passes to M2, and again is reflected back to the beam-splitter. Here, the two beams are recombined, being respectively transmitted and reflected to reach the detector D. Depending upon the displacement d from the position of equal path length M1', the movable mirror M2 introduces a path difference between the two routes, which results in constructive or destructive interference according to d.

[3 marks]

B2 contd.



path difference $= XYZ$ [1 mark]

$= X'YZ$ where X' is reflection of X in mirror M1

$= 2d\cos\theta$

$= s\cos\theta$ if $s =$ path difference at normal incidence [1 mark]

$\Rightarrow$ if incident light varies as $E(x) = E_0 \exp i(kx-\omega t)$, then the two components may be written $E(x_0)$, $E(x_0 + s\cos\theta)$

i.e. $E_0 \exp i(kx_0-\omega t)$, $E_0 \exp i(kx_0 - ks\cos\theta - \omega t)$

or, in terms of a common factor,

$$\underline{\underline{\begin{array}{l} E_0 \exp i(k(x_0-\tfrac{s}{2}\cos\theta - \omega t) \;\; \exp ik\tfrac{s}{2}\cos\theta \\ E_0 \exp i(k(x_0-\tfrac{s}{2}\cos\theta - \omega t) \;\; \exp -ik\tfrac{s}{2}\cos\theta \end{array}}}$$ [2 marks]

Thus the total electric field of the superposition will have the form

$$E_0 \exp i(k(x_0-\tfrac{s}{2}\cos\theta)-\omega t)) \left(e^{ik\frac{s}{2}\cos\theta} + e^{-ik\frac{s}{2}\cos\theta}\right)$$

$$= 2E_0 \exp i(k(x_0-\tfrac{s}{2}\cos\theta)-\omega t) \;\; \cos\left(\tfrac{ks}{2}\cos\theta\right)$$ [1 mark]

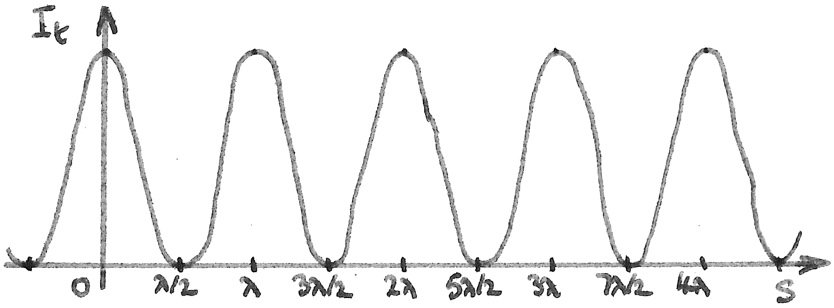
and the transmitted intensity, proportional to the squared magnitude of the field, will be proportional to $\left[\cos\left(\frac{ks}{2}\cos\theta\right)\right]^2$

i.e. $\underline{\underline{I_t \propto \cos^2\left(\frac{ks}{2}\cos\theta\right).}}$ [1 mark]

B2 contd. (a) single frequency He-Ne, $\lambda = 632.8\,\text{nm}$

$$\Rightarrow I_t \propto \cos^2 \frac{ks}{2} \quad \text{with } k = \frac{2\pi}{\lambda}.$$

* continuous sinusoidal variation
* period $= \lambda = 632.8\,\text{nm}$.



[3 marks]

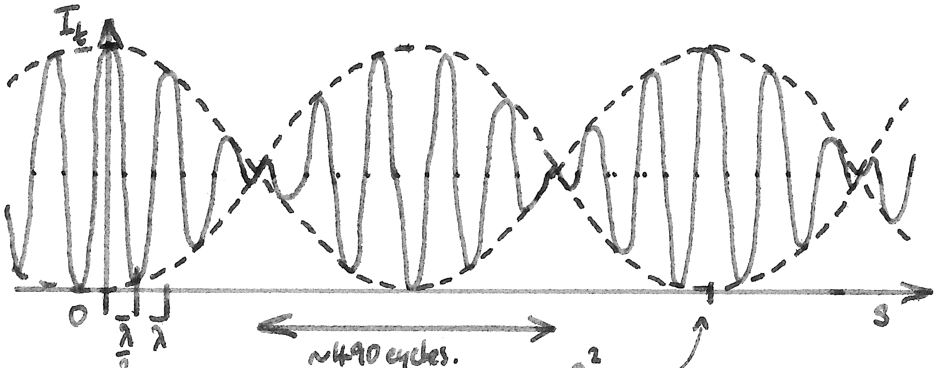
(b) With two wavelengths $\lambda_1 = 589.0\,\text{nm}$ and $\lambda_2 = 589.6\,\text{nm}$,

$$I_t \propto \cos^2 \frac{k_1 s}{2} + \cos^2 \frac{k_2 s}{2} \quad \text{where } k_{1,2} = \frac{2\pi}{\lambda_{1,2}}$$

$$\cos^2\theta \equiv \tfrac{1}{2}(1+\cos 2\theta)$$

$$\Rightarrow \quad I_t \propto 1 + (\cos k_1 s + \cos k_2 s)/2$$

$$= 1 + \cos \frac{k_1+k_2}{2} s \cos \frac{k_1-k_2}{2} s$$



$$\frac{\lambda^2}{2\delta\lambda} = 0.6\,\text{mm}$$

[3 marks]

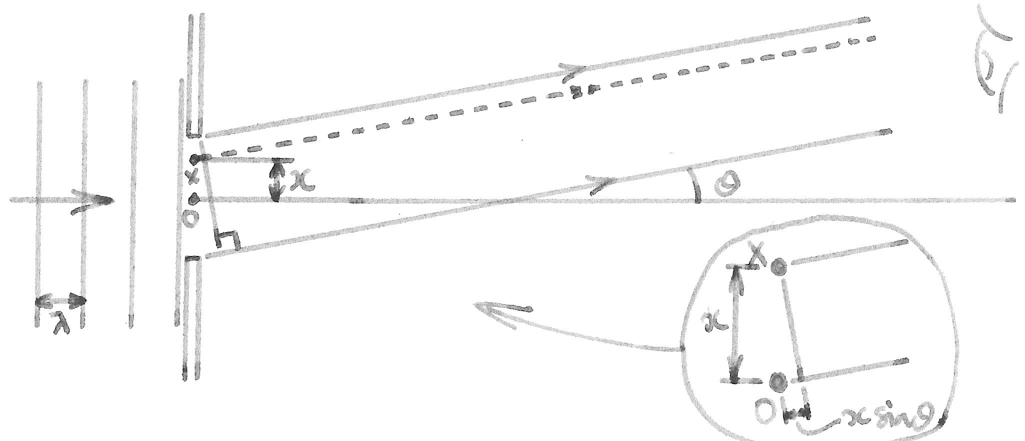
*Here drawn schematically: note real scales of two periods.

B2 contd. If r, t are not equal, the two paths nonetheless contribute equally to the output beam, since each path involves one reflection and one transit through the beamsplitter. The visibility and extinction of the fringe pattern observed is therefore unchanged. [2 marks]

Light not transmitted by the instrument is reflected back to the source. [1 mark]

B3. Fraunhofer diffraction is the effect upon a propagating wave of an opaque or refractive mask or obstruction, when viewed in the image plane of the wave source – for example, when plane waves are viewed from infinity. [2 marks]

Examples include the use of diffraction gratings for spectroscopy, the limit upon the resolution of an optical instrument due to the finite aperture of its optical elements, the vivid colours of butterfly wings, x-ray diffraction analysis, and the patterns apparent when a small, distant source is observed through a finely woven fabric. [2 marks]



We consider light, arriving at the slit as plane waves, passing via the point X a distance x from the slit centre, and proceeding at an angle θ to the distant observer. The path length differs from that through the slit centre O by $x \sin\theta$, corresponding to a phase difference

$$\delta\phi = \frac{2\pi}{\lambda} x \sin\theta$$ [1 mark]

The contribution to the diffracted amplitude is thus

$$a_0 \exp i\delta\phi = a_0 \exp i \frac{2\pi \sin\theta}{\lambda} x$$ [1 mark]

B3 contd. The total diffracted amplitude is found by summing contributions via all points in the slit

$$p(\theta) = \int_{-a/2}^{a/2} a_0 \exp i \frac{2\pi \sin\theta}{\lambda} x \, dx$$

$$= \frac{a_0}{i \frac{2\pi\sin\theta}{\lambda}} \left[\exp i \frac{2\pi\sin\theta}{\lambda} x \right]_{-a/2}^{a/2}$$

$$= \frac{2a_0}{2\pi\sin\theta/\lambda} \; \frac{\exp i \frac{2\pi\sin\theta}{\lambda}\frac{a}{2} - \exp -i \frac{2\pi\sin\theta}{\lambda}\frac{a}{2}}{2i}$$

$$= a_0 a \frac{\sin \frac{\pi a \sin\theta}{\lambda}}{\frac{\pi a \sin\theta}{\lambda}}.$$ [2 marks]

With two slits, the diffracted amplitude will be

$$p_2(\theta) = \int_{-\frac{d}{2}-\frac{a}{2}}^{-\frac{d}{2}+\frac{a}{2}} a_0 \exp i \frac{2\pi\sin\theta}{\lambda} x \, dx + \int_{\frac{d}{2}-\frac{a}{2}}^{\frac{d}{2}+\frac{a}{2}} a_0 \exp i \frac{2\pi\sin\theta}{\lambda} x \, dx$$ [1 mark]

$$= \int_{-a/2}^{a/2} a_0 \exp i \frac{2\pi\sin\theta}{\lambda}\left(x_1 - \frac{d}{2}\right) dx_1 + \int_{-a/2}^{a/2} a_0 \exp i \frac{2\pi\sin\theta}{\lambda}\left(x_2 + \frac{d}{2}\right) dx_2$$

$$= \left\{ \exp i \frac{2\pi\sin\theta}{\lambda}\left(\frac{-d}{2}\right) + \exp i \frac{2\pi\sin\theta}{\lambda}\left(\frac{d}{2}\right) \right\} p(\theta)$$ [1 mark]

$$= 2\, p(\theta) \cos \frac{\pi d \sin\theta}{\lambda}$$

B3 contd. Hence the diffracted intensity, proportional to the square of the diffracted amplitude, will be

$$I(\theta) = (2a_0 a)^2 \left\{ \frac{\sin \frac{\pi a \sin\theta}{\lambda}}{\frac{\pi a \sin\theta}{\lambda}} \cos \frac{\pi d \sin\theta}{\lambda} \right\}^2$$

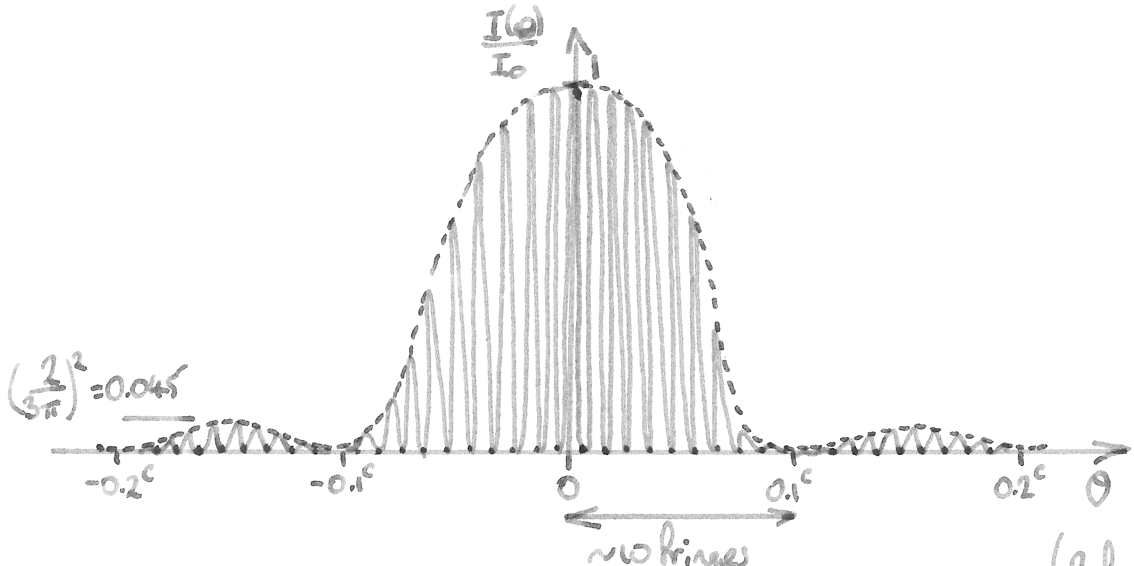
[2 marks]

as given.

If $a = 5\mu m$, $\lambda = 500nm$, $d \approx 10a$,

then $\beta = \frac{\pi a}{\lambda}\sin\theta = 10\pi\sin\theta \Rightarrow \sin\beta = 0$ when $\sin\theta = \frac{n\pi}{10\pi} = n/10$

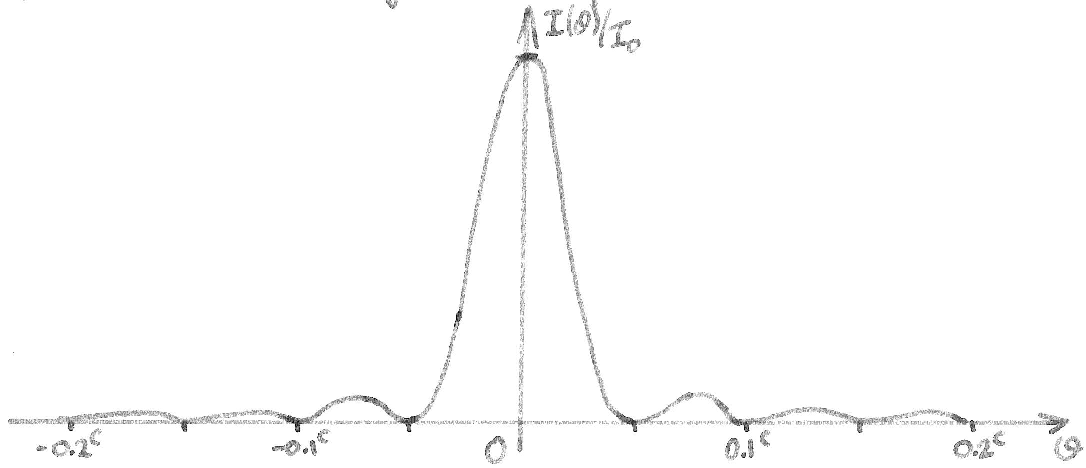
$\alpha/2 = \frac{\pi d}{\lambda}\sin\theta \approx 100\pi\sin\theta \Rightarrow \cos\frac{\alpha}{2} = 0$ when $\sin\theta = \frac{(n+\frac{1}{2})\pi}{100\pi} = \frac{n+\frac{1}{2}}{100}$



[4 marks]
(2 for sketch; 2 for labelling)

B3 cont'd. If $d = a$, then $\frac{\alpha}{2} = \frac{\pi d}{\lambda}\sin\theta = \frac{\pi a}{\lambda}\sin\theta = \beta$

$\Rightarrow$ pattern alternates between zeroes of $\sin\beta$ and $\cos\frac{\alpha}{2}$:

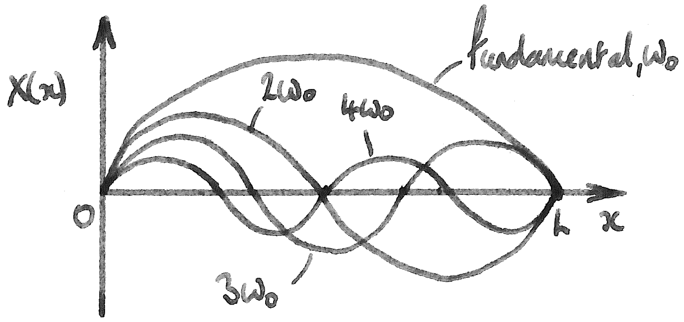


[2 marks]

This resembles – indeed, proves to be – $\left(\frac{\sin 2\beta}{2\beta}\right)^2 I_0$, a sinc^2 function with half the period, because two slits separated by their width between their centres are identical to a single slit of twice the width.

[2 marks]

B4.



[4 marks]
($\frac{1}{2}$ per curve)

(Shown are the magnitudes $X(x)$ of the wave displacements $y(x,t)$, where $y(x,t) = X(x)T(t)$. Vertical scaling is arbitrary.)

For a string of mass per unit length M, subject to a tension W, the wave speed $v = \sqrt{W/M}$ (for an inextensible string, small displacements, no friction etc.), and relates the frequency f and wavelength λ through [1 mark]

$$f = v/\lambda.$$ [1 mark]

From the above diagram, the standing wave wavelengths are given by $\lambda_n = 2L/n$ for the n^{th} harmonic. The frequencies will thus be

$$f_n = \frac{nv}{2L} = \underline{\underline{\frac{n}{2L}\sqrt{\frac{W}{M}}}} \quad \text{where } n = 1,2,3,4.$$ [1 mark]

The corresponding wavenumbers, $k = \frac{2\pi}{\lambda}$, will be

$$k_n = \frac{2\pi}{2L/n} = \underline{\underline{n\frac{\pi}{L}.}}$$ [1 mark]

*NB. W and M appear in the exam question as T and ρ, respectively.

B1 cont'd. If the harmonics have the same amplitudes a, they may be written as

$$X_n(x) = a \sin \frac{n\pi}{L} x$$ [1 mark]

The signal at the pick-up will therefore be proportional to

$$S_n = \sin n \frac{\pi d}{L} \quad \text{where } n = 1, 2, 3, 4.$$ [1 mark]

If $L = 648$mm and $d = 98$mm, the relative strengths of these first four harmonics will be

$$\begin{aligned} S_1 &= \sin \frac{\pi 98}{648} &= 0.457 \\ S_2 &= \sin 2 \frac{\pi 98}{648} &= 0.814 \\ S_3 &= \sin 3 \frac{\pi 98}{648} &= 0.989 \\ S_4 &= \sin 4 \frac{\pi 98}{648} &= 0.946 \end{aligned}$$ [2 marks]

The amplitude of the sinusoidal component with wavenumber k_n will be

$$a_n = \frac{2}{L} \int_0^L y(x) \sin n \frac{\pi}{L} x \, dx$$

which for the specific form shown is most conveniently split into two parts

$$= \frac{2}{L} \left\{ \int_0^{L/4} y(x) \sin \frac{n\pi x}{L} dx + \int_{L/4}^{L} y(x) \sin \frac{n\pi x}{L} dx \right\}$$ [1 mark]

and, as there is a node at $x = L$, we may start the second integral there – ie. $x' = L - x$:

$$a_n = \frac{2}{L} \left\{ \int_0^{L/4} y(x) \sin \frac{n\pi x}{L} dx + (-1)^{n-1} \int_0^{3L/4} y(x') \sin \frac{n\pi x'}{L} dx' \right\}$$ [1 mark]

B4 contd. Inserting the specific form of $y(x)$ now gives

$$a_n = \frac{2}{L}\left\{ \int_0^{L/4} \frac{4hx}{L} \sin\frac{n\pi x}{L}\,dx + (-1)^{n-1} \int_0^{3L/4} \frac{4hx'}{3L} \sin\frac{n\pi x'}{L}\,dx' \right\} \qquad [1\text{ mark}]$$

$$= \frac{2}{L}\left\{ \int_0^{n\pi/4} \frac{4h}{L}\left(\frac{L}{n\pi}\right)^2 \frac{n\pi x}{L} \sin\frac{n\pi x}{L}\, d\left(\frac{n\pi x}{L}\right) + (-1)^{n-1} \int_0^{3n\pi/4} \frac{4h}{3L}\left(\frac{L}{n\pi}\right)^2 \frac{n\pi x'}{L} \sin\frac{n\pi x'}{L}\, d\left(\frac{n\pi x'}{L}\right) \right\}$$

[1 mark]

which, using the result given, becomes

$$a_n = \frac{8h}{n^2\pi^2}\left\{ \sin\frac{n\pi}{4} - \frac{n\pi}{4}\cos\frac{n\pi}{4} + (-1)^{n-1}\frac{1}{3}\left[\sin\frac{3n\pi}{4} - \frac{3n\pi}{4}\cos\frac{3n\pi}{4}\right] \right\}$$

For the first four harmonics, we thus obtain

$$a_1 = \frac{8h}{\pi^2}\left(\frac{1}{\sqrt{2}} - \frac{\pi}{4}\frac{1}{\sqrt{2}} + \frac{1}{3}\left[\frac{1}{\sqrt{2}} + \frac{3\pi}{4}\frac{1}{\sqrt{2}}\right]\right) = \frac{4}{3}\frac{8h}{\pi^2}\frac{1}{\sqrt{2}} = 0.764h$$

$$a_2 = \frac{8h}{4\pi^2}\left(1 - \frac{\pi}{2}\cdot 0 - \frac{1}{3}\left[-1 - \frac{3\pi}{2}\cdot 0\right]\right) = \frac{4}{3}\frac{8h}{4\pi^2} = 0.270h$$

$$a_3 = \frac{8h}{9\pi^2}\left(\frac{1}{\sqrt{2}} + \frac{3\pi}{4}\frac{1}{\sqrt{2}} + \frac{1}{3}\left[\frac{1}{\sqrt{2}} - \frac{9\pi}{4}\frac{1}{\sqrt{2}}\right]\right) = \frac{4}{3}\frac{8h}{9\pi^2}\frac{1}{\sqrt{2}} = 0.085h$$

$$a_4 = \frac{8h}{16\pi^2}\left(0 + \pi - \frac{1}{3}\left[0 + 3\pi\right]\right) = 0 \qquad [2\text{ marks}]$$

The relative strengths in the electrical signal will thus be

$n=1$: $\frac{1}{\sqrt{2}} \sin\frac{98\pi}{648} = 0.350$

$n=2$: $\frac{1}{4} \sin 2\frac{98\pi}{648} = 0.220$

$n=3$: $\frac{1}{9\sqrt{2}} \sin 3\frac{98\pi}{648} = 0.084$

$n=4$: $0 = 0$

[2 marks]