WAVE PHYSICS - RESIT EXAM - SEPT. '09.

A1. Frequency = 2.45 GHz $\Rightarrow$ wavelength $= \frac{c}{2.45\times10^9} = \underline{\underline{0.12m.}}$ [1 mark]

The cavity will cause standing waves to build up, with a node-antinode distance of $\lambda/4 = 3$ cm. This will often be too far for heat conduction to be sufficient to distribute the heat evenly so, if a uniform heating rate is required, a turntable must be used. [1 mark] [2 marks]

A2. $$\hat{\omega} y = i\frac{\partial}{\partial t} a \exp i(kx-\omega t) = ia(-i\omega)\exp i(kx-\omega t)$$
$$= \omega\, a \exp i(kx-\omega t)$$
$$= \omega y$$

so by definition of operator, that $\hat{o}y = oy$, (or $o = \frac{\hat{o}y}{y}$), we see that the suggested form of $\hat{\omega}$ indeed yields the angular frequency $\hat{\omega}$. [2 marks]

The real sinusoidal wave $y = a\cos(kx-\omega t+\phi)$ may be written as a superposition of complex waves with positive and negative frequencies

$$a\cos(kx-\omega t+\phi) = \frac{a}{2}\Big(\exp i(kx-\omega t+\phi) + \exp i(-kx+\omega t-\phi)\Big).$$

The $\hat{\omega}$ operator will yield $+\omega$ and $-\omega$ for the two components, and hence zero for the superposition overall. [2 marks]

A3. Notable features of the signal are:

i) asymmetry about the origin ⇒ only sine terms in series, no cosine terms
ii) average over first half period is positive, over second half period negative
⇒ $\sin \omega_0 t$ coefficient positive
iii) average over first quarter period greater than average over second quarter, and average over third quarter period greater than average over fourth quarter
⇒ $\sin 2\omega_0 t$ coefficient positive

(i) and (iii) are sufficient to deduce that, of the four choices offered, only (a) can be correct.

Further checks may be made by calculating specific values. The plotted signal rises over a period that lasts a fifth of that during which it falls – ie. the peak is at 30° through the cycle, the minimum is at 330°, and there are zeroes at 0° and 180°. At these values we obtain

	0°	30°	180°	330°
a	0	0.87	0	−0.87
b	2.16	0.79	−0.27	0.79
c	0	0.18	0	−0.18
d	0.27	0.47	−2.16	0.47

Note that the approximation involved in considering only four terms leads to a small discrepancy between the above values for (a) and the sawtooth values – ie. 0.87 instead of 1.0. There will be further inaccuracies because the signal shown must be read by eye; but series (a) should still be the best candidate.

One could also check the slope at various points, by differentiating the candidate formulae. Candidate (b) is actually $\frac{d}{dt}(a)$, so the slope at the origin is 2.16 and at 180° is −0.27, with a ratio of −8.1. The full series would give $\frac{6}{\pi} = 1.91$, $\frac{-6}{5\pi} = -0.38$, ratio −5.

For correct answer [1 mark]
For rationale [3 marks]

A4. By substitution of $\xi = \xi_0 \cos(x - vt)$, or otherwise, we derive that the velocity v is given by $v^2 = E/\rho$. [1 mark]

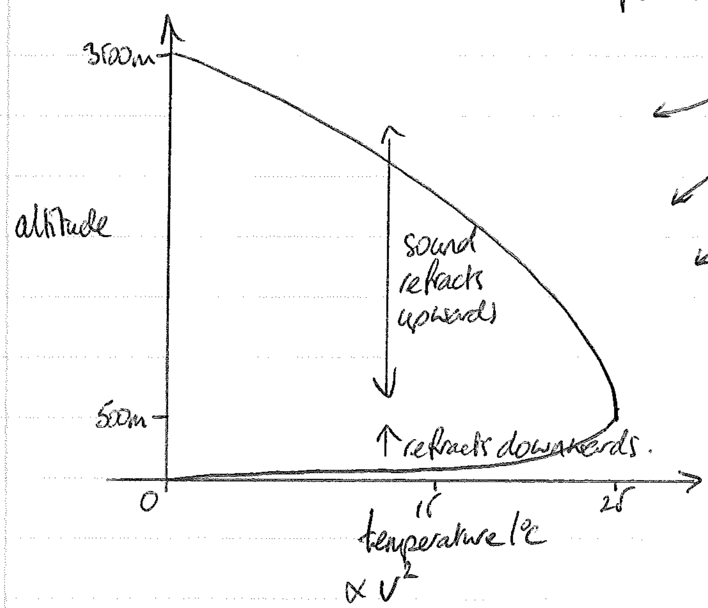
$$\Rightarrow \quad v = \sqrt{E/\rho} = \sqrt{\gamma P/\rho} = \sqrt{1.4 \times 1.013 \times 10^5 / 1.29}$$

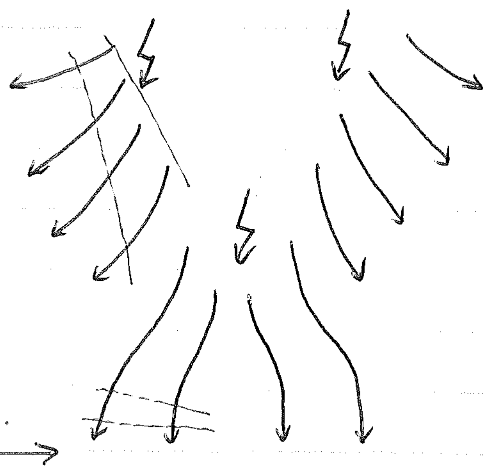
$$= \underline{\underline{\pm 332 \text{ m.s}^{-1}}}. \qquad [2 \text{ marks}]$$

Each kilometre therefore takes $\frac{1000}{332} = \underline{\underline{3.0 \text{ s.}}}$ [1 mark]

A5. $pV = RT \Rightarrow \rho = \frac{M}{V} = \frac{MP}{RT}$

$\Rightarrow v = \sqrt{\gamma P/\rho} = \underline{\underline{\sqrt{\frac{\gamma R T}{M}}}}$ i.e. independent of pressure, depends only upon the temperature [2 marks]





By considering refraction of sound wave fronts, we see that above 500m the sound will be refracted outwards and upwards; below, it will be refracted downwards.

[2 marks]

B1. Transverse and longitudinal wave motions define the wave as being manifest as a displacement of the medium perpendicular and parallel respectively to the direction of wave propagation [2 marks]

Transverse: light, radio waves, guitar strings
Longitudinal: sound, traffic density, thermal
Neither: quantum wavefunction [3 marks]

From Hooke's law for the element

$$T = \lim_{\delta x \to 0} (EA) \frac{\delta x + \delta \xi - \delta x}{\delta x} = EA \frac{\partial \xi}{\partial x}.$$ [2 marks]

Force on element, $T(x+\delta x) - T(x)$, $= \rho A \delta x \dfrac{\partial^2 \xi}{\partial t^2}$

$\Rightarrow$ rearranging and again taking the limit as $\delta x \to 0$,

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{1}{\rho A} \frac{\partial T}{\partial x} = \frac{EA}{\rho A} \frac{\partial}{\partial x} \frac{\partial \xi}{\partial x} = \frac{E}{\rho} \frac{\partial^2 \xi}{\partial x^2}$$ [2 marks]

Continuity conditions describe the physical constraints relating the wave function in the medium on one side of an interface to those on the other side. [1 mark]
For a guitar string, for example, the continuity of the string and requirement for finite accelerations of even infinitesimal sections lead to the continuity of the wave displacement and its spatial derivative. For sound waves, the medium displacement must again be continuous, but the other condition is continuity of pressure [1 mark]
A discontinuity in the material properties results in a change in wave speed and partial reflection of the incident wave [1 mark]

B1 cont'd. 1. Since the string is connected, under tension, to the can, and assuming that air cannot pass through the can base or leave a void against it, the air displacement must equal that of the can and string at x_0, i.e.

$$\xi_1(x_0,t) = \xi_2(x_0,t)$$ [1 mark]

2. If the mass of the can base can be neglected, then the mass of an element from $x_0 - \frac{\delta x}{2}$ to $x_0 + \frac{\delta x}{2}$ will vanish as $\delta x \to 0$. If the acceleration here is to remain finite, the net force on the element must be zero. We have derived this force for the general case earlier in the question, so here just apply it to the two regions:

$$E_1 A_1 \frac{\partial \xi_1}{\partial x}(x_0,t) = E_2 A_2 \frac{\partial \xi_2}{\partial x}(x_0,t)$$ [2 marks]

If all the sound energy is to be transmitted to the string, none can be reflected, so we need only consider the incident and transmitted waves, which we may write as

$$\xi_1(x,t) = f_1\left(t - \frac{x'}{V_1}\right)$$
$$\xi_2(x,t) = f_2\left(t - \frac{x'}{V_2}\right)$$

where for convenience we measure x' from $x=0$ at x_0.

Applying the first continuity condition gives

$$f_1(t) = f_2(t)$$ [1 mark]

Applying the second continuity condition gives

$$E_1 A_1 \left(\frac{-1}{V_1}\right) f_1' = E_2 A_2 \left(\frac{-1}{V_2}\right) f_2'$$ [1 mark]

where, from the result of the first condition, $f_1' = f_2'$

Hence, substituting $V = \sqrt{E/\rho}$,

$$E_1 A_1 \sqrt{\frac{\rho_1}{E_1}} = E_2 A_2 \sqrt{\frac{\rho_2}{E_2}}$$

$\Rightarrow$ $$A_1 \sqrt{E_1 \rho_1} = A_2 \sqrt{E_2 \rho_2} \quad \text{i.e.} \quad \underline{A_1 Z_1 = A_2 Z_2}$$ [1 mark]

B1 cont'd. Hence, for maximum transmission to the string,

$$A_1\sqrt{E_1\rho_1} = A_2\sqrt{E_2\rho_2} \quad \text{where} \quad \begin{aligned} E_1 &= 1.4\times10^5\ \mathrm{Nm^{-2}} \\ \rho_1 &= 1.29\ \mathrm{kg\,m^{-3}} \\ E_2 &= 7\times10^8\ \mathrm{Nm^{-2}} \\ \rho_2 &= 125\ \mathrm{kg\,m^{-3}} \\ A_2 &= \pi\left(\frac{1.5\,\mathrm{mm}}{2}\right)^2 \end{aligned}$$

$$A_1 = \pi\left(\frac{d}{2}\right)^2$$

$$\Rightarrow \quad d^2 = (1.5\,\mathrm{mm})^2\sqrt{\frac{7\times10^8 \cdot 125}{1.4\times10^5 \cdot 1.29}}$$

$$\Rightarrow \quad d = 1.5\,\mathrm{mm}\sqrt[4]{\frac{7\times10^8 \cdot 125}{1.4\times10^5 \cdot 1.29}} = 39.6\,\mathrm{mm} \underline{\underline{\approx 40\,\mathrm{mm}}} \text{ to appropriate precision}$$

[2 marks]

For curiosity, note that
(i) the edge of the base is likely to remain fixed, so the effective fraction of the diaphragm which moves will be about a half ($\Rightarrow$ "ideal" diameter ~ 5mm).
(ii) the mass of the base probably cannot be neglected
(iii) the base will move the air on the string side, too, although as this is unconstrained by the can the effect here will be less pronounced — closer to constant pressure assumed here
(iv) we've looked here for maximum coupling; in practice, one would wish for maximum amplitude, which might require a greater diameter
(v) we've neglected any flexural dynamics in the can base
(vi) waveguiding in the can may focus sound around the centre of the base, compensating for point (i).

B2. Travelling waves are those which maintain a constant form that is simply translated through space as time proceeds. Standing waves maintain a spatially fixed form that is multiplied by an evolving function of time [2 marks]

If $y(x,t) = f(u)$ where $u = x - ct$, then

$$\frac{\partial y}{\partial x} = \frac{dy}{du}\frac{\partial u}{\partial x} \; ; \; \frac{\partial^2 y}{\partial x^2} = \frac{d^2 y}{du^2}\left(\frac{\partial u}{\partial x}\right)^2 = \frac{d^2 y}{du^2}$$ [1 mark]

$$\frac{\partial y}{\partial t} = \frac{dy}{du}\frac{\partial u}{\partial t} \; ; \; \frac{\partial^2 y}{\partial t^2} = \frac{d^2 y}{du^2}\left(\frac{\partial u}{\partial t}\right)^2 = c^2 \frac{d^2 y}{du^2}$$ [1 mark]

$$\Rightarrow \quad c^2 \frac{d^2 y}{du^2} = A \frac{d^2 y}{du^2}$$

$\Rightarrow$ travelling waves of the form given are solutions to the wave equation provided [1 mark]
that $\underline{\underline{c^2 = A.}}$ [1 mark]

Boundary conditions describe the physical constraints upon a wave motion imposed by an external system or structure. [2 marks]
A guitar string is constrained to fixed points at the bridge and fret, limiting its oscillation frequency to a series of discrete harmonics.
A harbour wall prevents longitudinal motion of the water, causing the incoming wave to be reflected. [2 marks]

We seek solutions of the form $y(x,t) = f(u)$ where $u = x - ct$
which satisfy $y(vt, t) = a\cos\omega t$

$$\Rightarrow \quad f(u) = a\cos\omega t \quad \text{where } u = vt - ct = -(c-v)t$$ [1 mark]

$$\Rightarrow \quad \underline{\underline{f(u) = a\cos\omega\left(\frac{-u}{c-v}\right).}}$$ [1 mark]

$$\Rightarrow \quad y(x,t) = a\cos\omega\left(\frac{-x+ct}{c-v}\right) = \underline{\underline{a\cos\left(\frac{c\omega}{c-v}t - \frac{\omega}{c-v}x\right).}}$$ [1 mark]

$$\Rightarrow \text{Doppler shift} = \left|\omega - \frac{c\omega}{c-v}\right| = \underline{\underline{\frac{v}{c-v}\omega.}}$$ [1 mark]

B2 cont'd. At $x = vt + L$,

$$y(x,t) = a\cos\left(\frac{c}{c-v}\omega t - \frac{\omega}{c-v}(vt+L)\right)$$

$$= a\cos\left(\frac{c-v}{c-v}\omega t - \frac{\omega L}{c-v}\right)$$

$$= a\cos\left(\omega t - \frac{\omega L}{c-v}\right)$$ [2 marks]

This has the same frequency as the source, but differs in phase by a velocity-dependent term $\omega L/(c-v)$. Measurement of the phase of the received signal with respect to that of the source thus allows the velocity v to be determined. [2 marks]

For unambiguous interpretation of the measured phase, we need the phase shift to vary by less than one cycle over the range of speeds encountered.

i.e. $$\left|\frac{\omega L}{c-v_{max}} - \frac{\omega L}{c-v_{min}}\right| < 2\pi$$

If negative velocities are allowed, we set $v_{max} = +10\,m.s^{-1}$, $v_{min} = -10\,m.s^{-1}$, recall that $c = 1600\,m.s^{-1}$ and hence obtain

$$\omega L < 2\pi\frac{(c-v_{min})(c-v_{max})}{v_{max}-v_{min}} = 2\pi\frac{1600.1600}{20} \approx 2\pi\frac{1600^2}{20}$$

$$= 1.12\times10^5\times2\pi\ m.rads.sec^{-1}$$

$\Rightarrow$ $fL < 1.1\times10^5\ Hz$ [1 mark]

For maximum sensitivity, though, operation should be close to this limit.

e.g. $L = 10\,cm$, $f = 1\,MHz$. [1 mark]

(or any other satisfactory choice.)

B3. The Huygens description allows the propagation of a wavefront to be determined by placing imaginary sources along a given wavefront and calculating the disturbance that would result some time later from those sources alone. [2 marks]

This allows diffraction to be addressed for, when an object partially obstructs a wavefront preventing its simple evolution to be assumed, a series of hypothetical sources placed along the unobstructed parts of the wavefront will simulate the subsequent propagation. [2 marks]

A rectangular / top-hat mask of width w produces a sinc-function diffraction pattern with the first minimum at $\underline{\theta = \lambda/w}$. Taking 2θ as the 'divergence' of the beam gives [2 marks]

$$w = 5\mu m : \text{divergence} = \frac{2.780nm}{5\mu m} = 0.32^c = \underline{\underline{18^\circ}}$$

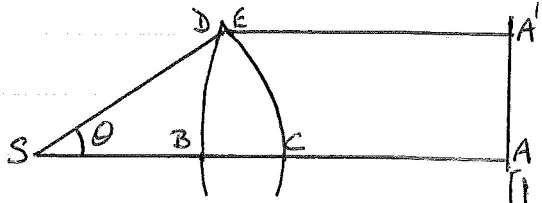
$$w = 10\mu m : \text{divergence} = \frac{2.780nm}{10\mu m} = 0.16^c = \underline{\underline{9^\circ}}. \quad [2 \text{ marks}]$$

(One could equally – or perhaps better – deal with the FWHM etc.)

(a) Lens diameter $= 2 \times \text{focal length} \times \tan\left(\frac{\text{divergence}}{2}\right)$

$$= 2 \times 10mm \times \tan 9^\circ = \underline{\underline{3.2mm}} \quad [1 \text{ mark}]$$

(b) The 'optical paths' must be equal for routes SBCA and SDEA'. The difference in lens thickness is BC–DE. [1 mark]



$\Rightarrow SD + DE\eta + EA' = SB + BC\eta + CA$

Note that

$$SD + DE + EA' \simeq SB + BC + CA \begin{cases} + SD(1-\cos\theta) \\ \text{OR } SB\left(\frac{1}{\cos\theta} - 1\right) \end{cases}$$

where $SB \equiv f$, the focal length of the lens. [1 mark]

$$\Rightarrow (DE - BC)(\eta - 1) = -f\left(\frac{1}{\cos\theta} - 1\right)$$

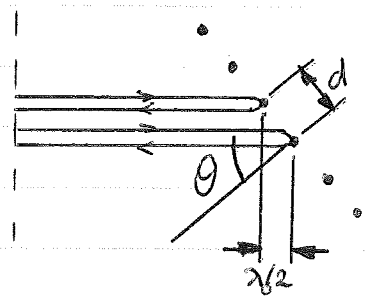
B3 cont'd $\Rightarrow$ difference in thickness, BC-DE $= \dfrac{f\left(\frac{1}{\cos\theta} - 1\right)}{n-1}$

$$= \frac{10\text{mm}\left(\frac{1}{\cos 9^\circ} - 1\right)}{1.53 - 1}$$

$$= 0.24\text{mm}.$$ [1 mark]

paths shown differ in total length by a single wavelength λ_0

$\Rightarrow \sin\theta = \dfrac{\lambda_0/2}{d} = \dfrac{\lambda_0}{2d}.$



[3 marks]

For incident and diffracted angles θ and $\phi = \theta + \delta\theta$,

$d(\sin\theta + \sin\phi) = \lambda$ [1 mark]

$\Rightarrow d(\sin\theta + \sin(\theta + \delta\theta)) = \lambda$

$\Rightarrow \sin\theta + \sin\theta\cos\delta\theta + \cos\theta\sin\delta\theta = \lambda/d$

$\Rightarrow$ taking $\delta\theta$ to be small, as the difference in wavelength is slight

$2\sin\theta + \cos\theta\,\delta\theta = \lambda/d$

$\Rightarrow \delta\theta = \dfrac{1}{\cos\theta}\left(\dfrac{\lambda}{d} - 2\sin\theta\right) = \dfrac{1}{\cos\theta}\left(\dfrac{\lambda}{d} - \dfrac{\lambda_0}{d}\right) = \dfrac{\delta\lambda}{d\cos\theta}.$ [2 marks]

To miss the gain region, we need $\delta\theta > \dfrac{5\text{um}}{10\text{mm}}$, the angle subtended by the gain region at the lens

$\Rightarrow \delta\lambda = d\cos\theta\,\delta\theta = \dfrac{\lambda_0}{2\sin\theta}\cos\theta\,\dfrac{5\text{um}}{10\text{mm}} = \dfrac{780\times10^{-9}\text{m}}{2\cancel{\sin\theta}}\cancel{\cos\theta}\,\dfrac{5\text{um}}{10\text{mm}} = 0.2\text{nm}.$ [2 marks]

B4. Dispersion describes the spreading of a wavepacket as it propagates, and is associated with a variation in the phase velocity with the frequency of sinusoidal components of the wavepacket. [2 marks]

Practical consequences include the separation of light into its spectrum by a prism, chromatic aberration in imaging lenses, pulse distortion in optical fibres and, as we'll see below, the evolution of a quantum wavepacket in accordance with the uncertainty principle [2 marks]

If $y(x,t) = f(u)$ where $u = x - ct$, then

$$\frac{\partial y}{\partial t} = \frac{dy}{du}\frac{\partial u}{\partial t} = -c\frac{dy}{du} \equiv -cf' \qquad \text{[1 mark]}$$

$$\frac{\partial y}{\partial x} = \frac{dy}{du}\frac{\partial u}{\partial x} = \frac{dy}{du} \equiv f'$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{d^2y}{du^2}\left(\frac{\partial u}{\partial x}\right)^2 = \frac{d^2y}{du^2} \equiv f'' \qquad \text{[1 mark]}$$

so for $y(x,t) = f(u)$ to solve the Schrödinger equation,

$$\alpha(-c)f' = \beta f'' + \gamma f \qquad \text{[1 mark]}$$

which will not generally be true for arbitrary x, t. [1 mark]

If $y(x,t) = y_0 \exp i(kx - \omega t)$, then to solve the Schrödinger equation,

$$\alpha(-i\omega)\, y_0 \exp i(kx-\omega t) = \beta(-k^2) y_0 \exp i(kx-\omega t) + \gamma y_0 \exp i(kx-\omega t) \qquad \text{[1 mark]}$$

$$\Rightarrow \quad -i\omega\alpha = -k^2\beta + \gamma \qquad \cancel{\text{[1 mark]}}$$

so the given forms are indeed possible solutions provided that $-i\omega\alpha = -k^2\beta + \gamma$. [1 mark]

B4 cont'd Inserting the specific values of α, β, γ, we obtain

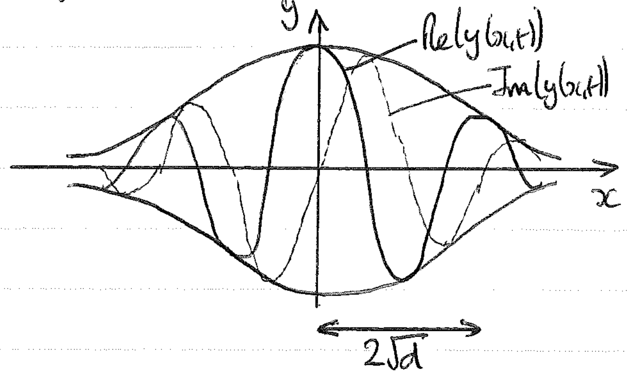
$$-i\omega i\hbar = -k^2 \frac{-\hbar^2}{2m} + V$$

$$\Rightarrow \quad \underline{\underline{\hbar\omega = \frac{\hbar^2 k^2}{2m} + V}} \qquad \Rightarrow \quad \omega = \frac{\hbar k^2}{2m} + V/\hbar.$$ [1 mark]

$$\Rightarrow \text{if } V=0, \quad v_p = \omega/k = \underline{\underline{\frac{\hbar k}{2m}}}.$$ [1 mark]

At $t=0$ we have $y(x,t) = \sqrt{\frac{\pi}{d}}\, e^{i(kx-\omega t)}\, e^{-x^2/4d}$

so the wavepacket is a Gaussian-modulated travelling complex wave of width $2\sqrt{d}$ centred on the origin.



[2 marks]

The wavepacket evolves by
- moving in the +ve x-direction with speed V
- increasing in width and reducing in height
- the complex wave factor moving in the +ve x-direction with speed ω/k. [2 marks]

B4 cont'd. For the wavepacket given,

$$\frac{\partial y}{\partial t} = \sqrt{\frac{\pi}{d+at}}\, e^{i(kx-\omega t)} \exp - \frac{(x-vt)^2}{4(d+at)} \left\{ \frac{-2(x-vt)(-v)}{4(d+at)} + \frac{a(x-vt)^2}{4(d+at)^2} - i\omega \right\}$$

$$+ e^{i(kx-\omega t)} \exp - \frac{(x-vt)^2}{4(d+at)} \sqrt{\pi} \left(\frac{-1}{2}\right) a\, (d+at)^{-3/2}$$

$$= \sqrt{\frac{\pi}{d+at}}\, e^{i(kx-\omega t)} \exp - \frac{(x-vt)^2}{4(d+at)} \left\{ \frac{2v(x-vt)}{4(d+at)} + \frac{a(x-vt)^2}{4(d+at)^2} - i\omega - \frac{a}{2(d+at)} \right\}$$ [1 mark]

$$\frac{\partial y}{\partial x} = \sqrt{\frac{\pi}{d+at}}\, e^{i(kx-\omega t)} \exp - \frac{(x-vt)^2}{4(d+at)} \left\{ \frac{-2(x-vt)}{4(d+at)} + ik \right\}$$

$$\Rightarrow \frac{\partial^2 y}{\partial x^2} = \sqrt{\frac{\pi}{d+at}}\, e^{i(kx-\omega t)} \exp - \frac{(x-vt)^2}{4(d+at)} \left\{ \left(\frac{-2(x-vt)}{4(d+at)} + ik \right)^2 - \frac{2}{4(d+at)} \right\}$$ [1 mark]

so, substituting into the Schrödinger equation and cancelling common terms,

$$\alpha \left\{ \frac{2v(x-vt)}{4(d+at)} + \frac{a(x-vt)^2}{4(d+at)^2} - i\omega - \frac{a}{2(d+at)} \right\} = \beta \left\{ \left(\frac{-2(x-vt)}{4(d+at)} + ik \right)^2 - \frac{2}{4(d+at)} \right\} + \gamma$$

Collecting terms:

$$\frac{\alpha a (x-vt)^2}{4(d+at)^2} = \frac{4\beta(x-vt)^2}{16(d+at)^2} \qquad \Rightarrow \alpha a = \beta, \text{ correct from definition of } a$$

$$\frac{2\alpha v(x-vt)}{4(d+at)} = \frac{-4i\beta k(x-vt)}{4(d+at)} \qquad \Rightarrow \alpha v = -2ik\beta$$

$$\Rightarrow \underline{\underline{v = -2ik \frac{i\hbar}{2m} = \frac{\hbar k}{m}.}}$$

$$-i\alpha\omega - \frac{a\alpha}{2(d+at)} = -\beta k^2 - \frac{2\beta}{4(d+at)} + \gamma$$

$$\Rightarrow -i\alpha\omega = -\beta k^2 + \gamma \quad \text{and} \quad \frac{a\alpha}{2(d+at)} = \frac{\beta}{2(d+at)}$$

$$\Rightarrow \hbar\omega = \frac{\hbar^2 k^2}{2m} + V \quad \text{as before, and } a\alpha = \beta, \text{ as above}$$

So the wavepacket is indeed a solution to the Schrödinger equation, provided that $v = \hbar k/m$. [2 marks]