SEMESTER 1 EXAMINATION 2016-2017

WAVE PHYSICS

Duration: 120 MINS (2 hours)

This paper contains 9 questions.

Answers to Section A and Section B must be in separate answer books.

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language dictionary is permitted ONLY IF it is a paper version of a direct 'Word to Word' translation dictionary AND it contains no notes, additions or annotations.

18 page examination paper.

Section A

A1. The equation of a sinusoidal transverse wave travelling along a string is

$$\psi(x,t) = A\sin(\phi(x,t)) = A\sin(5\pi x - 40\pi t),$$

where $A = (10/\pi)$ m, x and ψ are measured in metres and t is in seconds.

(a) Find the wavelength and frequency of the wave. [1]

By comparison with the standard form $\psi = A\sin(kx - \omega t)$ where $k \equiv 2\pi/\lambda$ and $\omega \equiv 2\pi f$, λ being the wavelength and f the frequency, we obtain [0.5 for each]

$$\begin{aligned} \textit{wavelength}\quad \lambda &= 2\pi/(5\pi) = 0.4\,\mathrm{m} \\ \textit{frequency}\quad f &= 40\pi/(2\pi) = 20\,\mathrm{Hz} \end{aligned}$$

[1]

(b) Find the maximum *transverse* speed of any particle in the string. [1]

The transverse speed will be

$$\frac{\partial\psi}{\partial t} = -40\pi A\cos(5\pi x - 40\pi t)$$

which has a maximum value of $40\pi A = 40\pi \times (10/\pi)\,\mathrm{m\,s^{-1}} = 400\,\mathrm{m\,s^{-1}}$. *[1]*

(c) By writing the phase $\phi(x,t)$ in terms of the angular frequency ω and wavenumber k, or otherwise, show that the *phase velocity* v_p of a point of constant phase ϕ is given by $v_\mathrm{p} = \omega/k$. [2]

The path of a point of constant phase will be defined by [1]

$$\phi(x,t) = kx - \omega t = \phi_0$$

where ϕ_0 is a constant. We may either differentiate this expression with respect to time, giving

$$k\frac{\mathrm{d}x}{\mathrm{d}t} - \omega = 0$$

whence $\mathrm{d}x/\mathrm{d}t = \omega/k$, or rearrange it to give an equation of motion

$$x = \phi_0/k + (\omega/k)t$$

from which the wave propagation speed may be identified as ω/k. [1] *[2]*

A2. What is meant by *dispersion*? [1]

Dispersion is the variation of wave propagation speed with the frequency of sinusoidal component, causing wavepackets to spread as they propagate. *[1]*

Give an expression for the *group velocity* of a wave whose angular frequency is given as a function of the wavenumber k by $\omega(k)$. [1]

The group velocity v_g by

$$v_\mathrm{g} = \frac{\partial \omega}{\partial k}.$$ *[1]*

Determine the relationship between the phase velocity $v_\mathrm{p} = \omega/k$ and the group velocity for shallow water capillary waves whose dispersion relation is

$$\omega^2 = \frac{h_0 \sigma}{\rho} k^4$$

where σ is the surface tension, h_0 the depth of the water, and ρ its density. [2]

For capillary waves in shallow water, $\omega^2 = (h_0\sigma/\rho)\,k^4$, so [0.5]

$$v_\mathrm{p} = \frac{\sqrt{(h_0\sigma/\rho)}\,k^2}{k} = \sqrt{(h_0\sigma/\rho)}\,k$$

and [1]

$$2\omega v_\mathrm{g} = 2\omega\frac{\partial \omega}{\partial k} = \frac{\partial}{\partial k}\left[(h_0\sigma/\rho)\,k^4\right] = (h_0\sigma/\rho)\,4k^3$$

whence

$$v_\mathrm{g} = \frac{(h_0\sigma/\rho)\,4k^3}{2\sqrt{(h_0\sigma/\rho)}\,k^2} = 2\sqrt{(h_0\sigma/\rho)}\,k$$

hence the group velocity is twice the phase velocity [0.5]. *[2]*

A3. Explain the meanings of *transverse* and *longitudinal* wave motions, and give an example of each. [3]

Transverse waves involve the propagation of a particle or medium displacement, or of a field component, in a direction perpendicular to the direction of wave propagation [1]. Longitudinal waves involve the propagation of a displacement or field component parallel to the direction of propagation [1]. Examples of transverse waves include electromagnetic radiation, gravitational waves, guitar strings and shallow

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water waves [0.5]; longitudinal waves include sound and thermal waves [0.5]. (Note that longitudinal components of electromagnetic, gravitational and guitar waves are also possible, and that shallow water waves involve longitudinal motion in their propagation.) *[3]*

Give an example of a wave that is neither transverse nor longitudinal. [1]

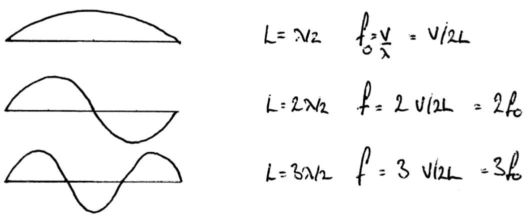
Waves that are neither transverse nor longitudinal include quantum wavefunctions, combustion/flame wavefronts, chemical waves, biochemical/neurological waves (e.g. cephalopods), Mexican waves (perhaps), even waves of fear [1]. *[1]*

A4. Explain, with examples, the *boundary conditions* that may apply to a wave motion. [2]

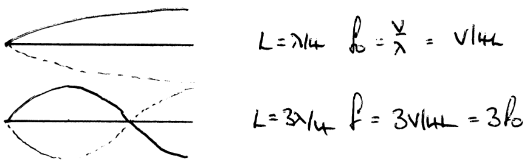
Boundary conditions are constraints imposed upon the wave at particular positions by the presence of external influences [1]. For example, a guitar string is constrained at the bridge and fret to have a fixed (zero) displacement; the air column of a clarinet must have at its open end the unperturbed atmospheric pressure; shallow water waves at a harbour wall may move vertically but not horizontally [1]. *[2]*

By considering the different boundary conditions governing the instruments, explain why violin strings can support all harmonics of the fundamental frequency, but instruments like the clarinet produce only odd-numbered harmonics. [2]

A guitar string is fixed at both ends, and therefore supports sinusoidal standing waves for which the string length is an integer number of half-wavelengths. The corresponding frequencies are therefore integer multiples of the fundamental [1].



The clarinet is closed at the mouthpiece end (displacement = 0) and open to atmospheric pressure at the bell (pressure variation = 0). The pressure depends not upon the displacement but upon its spatial derivative - i.e. compression rather than simple movement of a given element of the air column. Nodes in the pressure variation hence coincide with anti-nodes of displacement.



Hence the length will be an odd number of quarter wavelengths, so only odd harmonics are allowed [1]. *[2]*

A5. Outline how the *mean* frequency, and the *standard deviation* of the frequency, may be determined for a complex wave motion,

(a) from its expression as a function of time, $\psi(t)$, and [2]

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The mean and standard deviation of the frequency may be obtained by application of the frequency operator $\hat{\omega} \equiv \mathrm{i}\partial/\partial t$, and weighting the results according to the instantaneous intensity, i.e., for the mean frequency

$$\langle\omega\rangle = \int \psi(t)^* \hat{\omega}\psi(t)\,\mathrm{d}t = \int \psi(t)^* \frac{\partial\psi(t)}{\partial t}\mathrm{d}t, \qquad [\,1\,]$$

and for the standard deviation

$$\sigma_\omega^2 = \int \left[\hat{\omega}\psi(t) - \langle\omega\rangle\psi(t)\right]^2 \mathrm{d}t = \int \left[\frac{\partial\psi(t)}{\partial t} - \langle\omega\rangle\psi(t)\right]^2 \mathrm{d}t. \qquad [\,1\,]$$

[Formulae are not necessarily required if a good textual explanation is given.]

(b) from its frequency spectrum, $g(\omega)$. [2]

If a wave motion contains components with a range of frequencies, then the mean frequency may be obtained from the spectrum (Fourier transform) by calculating the mean value of the frequency, weighted according to the intensity of each component — that is

$$\langle\omega\rangle = \int |g(\omega)|^2\,\omega\,\mathrm{d}\omega. \qquad [\,1\,]$$

The standard deviation may be similarly determined by calculating

$$\sigma_\omega^2 = \int |g(\omega)|^2\,(\omega - \langle\omega\rangle)^2\,\mathrm{d}\omega. \qquad [\,1\,]$$

Section B

B1. (a) Explain briefly the phenomenon of wave *interference*, and what is meant by *constructive* and *destructive* interference. [4]

Interference is the phenomenon by which, when two or more waves are added, the resultant intensity differs from the simple sum of the component intensities [1]. When waves are superposed, it is their amplitudes that are added, and the intensities are proportional to the squares of the amplitudes [1]. The nature of the interference depends upon the relative phases of the interfering wave components [1]. Two in-phase waves hence interfere constructively, resulting in a greater overall intensity than the total of the components, whereas out-of-phase components interfere destructively, giving a combined intensity that is less than the component total [1]. *[4]*

(b) Outline the *Huygens description* of wave propagation, and explain how it can be used to calculate the diffraction pattern of an illuminated object. [4]

The Huygens description allows the propagation of a wavefront to be determined by placing imaginary sources along a given wavefront and calculating the disturbance that would result some time later from those sources alone. When performed geometrically, the new wavefront lies along the common tangent to the circular wavefronts from adjacent contributions. *[2]*

Imaginary sources are placed along a wavefront as it encounters the diffracting object, and the disturbance radiated by each source is considered to be modulated according to the transmission of the object at that point. The diffracted wave is then the resultant of the transmitted amplitudes. *[2]*

The light of a collimated laser beam of wavelength λ falls at normal incidence upon a flat screen. Between the screen and the laser, in the middle of the beam at a distance d from the screen, the beam is scattered from a small particle.

(c) Show that, at the screen, the interference between the scattered light and the unperturbed laser beam will result in a series of bright rings whose radii are given, for integer values of n, by

$$r_n^2 = n\left(n + \frac{2d}{\lambda}\right)\lambda^2 .$$ [4]

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For constructive interference, the path difference between the laser beam and the scattered light will be an integer number of wavelengths [1], i.e., [1]

$$\sqrt{r^2 + d^2} - d = n\lambda$$

hence [2]

$$r^2 = (n\lambda + d)^2 - d^2 = n^2\lambda^2 + 2n\lambda d = n\left(n + \frac{2d}{\lambda}\right)\lambda^2.$$ *[4]*

Photochemical etching techniques are used to make the screen transparent at the positions of the interference maxima, and the scattering particle is removed.

(d) Show that, when illuminated with the original collimated laser beam, the light transmitted by the etched screen comprises three components:

(i) the plane wavefronts of the original illumination [1]

(ii) curved wavefronts that converge upon a point a distance d after the screen; and [2]

(iii) curved wavefronts that appear to diverge from a point a distance d before the screen. [2]

Since the distances from all the transmissive regions to a plane parallel to the screen are the same, secondary wavelets from the transmissive regions will, as usual, reconstruct a plane wavefront that continues the incident illumination with reduced intensity. *[1]*

Since the transmissive regions were constructed to lie at the positions from which the paths to a point d from the screen differ by integer numbers of wavelengths, the Huygens wavefronts will arrive in phase at a point d after the screen. [1] At their different transverse positions before and after this point, they will be equidistant from this focus, and will hence form spherical wavefronts that converge towards or diverge from it. [1] The nth ring contributes a wavefront that is n wavelengths behind that contributed by transmission through the centre of the screen. *[2]*

The same geometry shows that the nth ring can also form part of a spherical wavefront that is n wavelengths ahead *of that transmitted through the centre, and appears to spread from the position of the original scattering particle, a distance d before the screen.* *[2]*

(e) Explain how the positions of the points to and from which the curved wavefronts convergence and divergence will vary if the wavelength of the illuminating laser is changed, assuming that $n \ll 2d/\lambda$. [2]

An approximate or qualitative solution may be obtained by considering small sections of the zone plate to act as diffraction gratings, and determining where the rays diffracted from different radii intersect.

A more formal solution requires that the path distances from the transmissive rings should be integer multiples of the new wavelength λ' [1], so that

$$r_n^2 + d'^2 = n\left(n + \frac{2d}{\lambda}\right)\lambda^2 + d'^2 = (n\lambda' + d')^2$$

where d' is the distance to the convergence/divergence point at a new wavelength λ'. If $n \ll 2d/\lambda, 2d'/\lambda'$, this gives

$$2dn\lambda = 2nd'\lambda'$$

and hence $d' = (\lambda/\lambda')d$. [1] *[2]*

(f) Suggest an optical application for this etched screen. [1]

The mask acts as a lens [1] of focal length $\pm d$; the focal length changes if the illumination wavelength is changed, but the lens is thinner than a refractive lens. Such zone plates *are hence used as compact single-wavelength lenses; they have also been investigated for use as the basis for bifocal contact lenses.* *[1]*

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B2. (a) What is meant by the *Fourier transform*? How may it be defined mathematically? [4]

The Fourier transform allows a function of time or position to be instead represented as a function of frequency or spatial frequency – i.e., by the spectrum of sinusoidal or complex exponential components into which it may be resolved. *[2]*

The component with a given frequency is obtained by multiplying the function by a sine wave (or complex exponential wave) with the same (or negative) frequency, and integrating over the range of the function, e.g.

$$\begin{aligned} f(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}t \\ f(\omega) &= \int_{-\infty}^{\infty} f(t) \cos \omega t \, \mathrm{d}t. \end{aligned}$$

(The overall factor of $1/\sqrt{2\pi}$ is a somewhat arbitrary choice, determined by whether the aim is to symmetrize the Fourier transform and its inverse or to normalize the spectral intensity as a function of frequency or angular frequency.) *[2]*

A yacht's pulsed radar emits microwave bursts of duration T of a single frequency $f_0 = \omega_0/2\pi$ (where $\omega_0 T \gg 1$), so that the wave amplitude is

$$a(t) = \begin{cases} \sin \omega_0 t & (-T/2 \le t \le T/2)\,; \\ 0 & (t < -T/2,\ t > T/2)\,. \end{cases}$$

(b) Show that the Fourier transform of a single burst is given by

$$b(\omega) \propto \frac{\sin\{(\omega_0 - \omega)T/2\}}{\omega_0 - \omega} - \frac{\sin\{(\omega_0 + \omega)T/2\}}{\omega_0 + \omega}$$

and show that, for frequencies around ω_0, the second term may be neglected. [4]

The antisymmetry of $a(t)$ allows it to be expressed as a superposition of sine waves $\sin \omega t$ of amplitudes $b(\omega)$, where

$$\begin{aligned} b(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(t) \sin \omega t \, \mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}} \int_{-T/2}^{T/2} \sin \omega_0 t \sin \omega t \, \mathrm{d}t & [\,1\,] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-T/2}^{T/2} \frac{1}{2} [\cos(\omega_0 - \omega)t - \cos(\omega_0 + \omega)t]\, \mathrm{d}t & [\,1\,] \end{aligned}$$

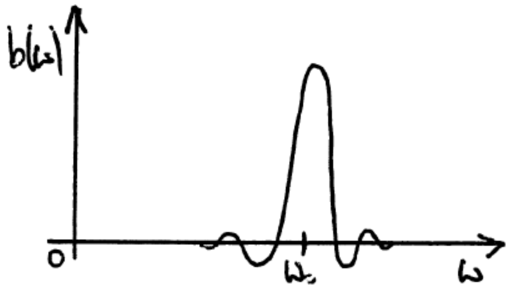
$$= \frac{1}{2\sqrt{2\pi}} \left[\frac{\sin(\omega_0 - \omega)t}{\omega_0 - \omega} - \frac{\sin(\omega_0 + \omega)t}{\omega_0 + \omega} \right]_{-T/2}^{T/2}$$

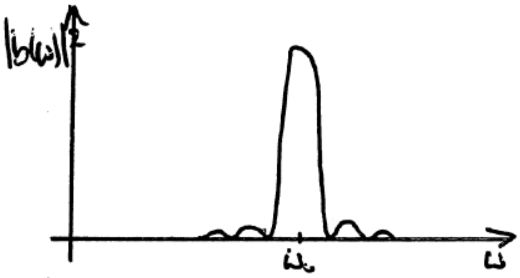
$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(\omega_0 - \omega)T/2}{\omega_0 - \omega} - \frac{\sin(\omega_0 + \omega)T/2}{\omega_0 + \omega} \right]$$ *[1]*

Since $\omega_0 T \gg 1$, for frequencies around $\omega = \omega_0$ the first term will dominate, since well away from the principal maximum it is the denominators that are important, so

$$b(\omega) \approx \frac{T/2}{\sqrt{2\pi}} \left[\frac{\sin(\omega_0 - \omega)T/2}{(\omega_0 - \omega)T/2} \right].$$ *[1]*

(c) Sketch the amplitude and intensity spectra corresponding to your result for frequencies around $\omega = \omega_0$, labelling any significant frequencies. [4]





where the zeroes occur at $\omega = \omega_0 + 2n\pi/T$. [2 marks each] *[4]*

(d) Given that $[\sin(a)/a]^2 = 1/2$ when $a = 1.392$, derive the full width at half-maximum intensity of the spectrum of each burst. [4]

At half-maximum intensity, $|b(\omega)|^2 = (1/2)\,|b(\omega_0)|^2$, so

$$\left| \frac{\sin(\omega_0 - \omega)T/2}{(\omega_0 - \omega)T/2} \right|^2 = \frac{1}{2}$$ *[1]*

i.e. $[\sin(a)/a]^2 = 1/2$ where $a \equiv \pm(\omega_0 - \omega)T/2$, so

$$\frac{(\omega_0 - \omega)T}{2} = \pm 1.392$$ *[1]*

$$\Rightarrow |\omega_0 - \omega| = \frac{2 \times 1.392}{T}.$$ *[1]*

The full width at half-maximum is hence

$$FWHM = 2\frac{2 \times 1.392}{T} = \frac{5.568}{T}\ \mathrm{rad\,s^{-1}} \equiv \frac{0.886}{T}\ \mathrm{Hz}.$$ *[1]*

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(e) The radar, which operates by reflecting microwave bursts from other vessels or obstacles, is required to measure the positions of the reflecting objects with a precision of 30 m. Taking the speed of electromagnetic waves to be 3×10^8 m s^{-1}, estimate the maximum allowable duration of each radar burst, and hence estimate the minimum bandwidth of the transducers used to generate it. [4]

The difference in delays between pulses reflected from objects differing in distance by $\Delta R = 30$ m will be $2\Delta R/c$, where c is the speed of propagation. To resolve these objects, it seems reasonable that the pulse length T should be no larger than this, i.e.

$$T \le \frac{2\Delta R}{c}. \qquad (= 200\,\mathrm{ns})$$ *[2]*

The spectrum of the pulse will therefore satisfy

$$\mathit{FWHM} = \frac{0.886}{T} \ge \frac{0.886\,c}{2\Delta R}.$$ *[1]*

If $c = 3 \times 10^8$ m s^{-1} and $\Delta R = 30$ m, we thus obtain

$$\mathit{FWHM} \ge 4.4\,\mathrm{MHz}.$$ *[1]*

For negligible pulse distortion, the microwave transducers must have a bandwidth of at least around this value. There are several alternative criteria, e.g. requiring a certain steepness to the pulse edge, which will give answers of this order of magnitude.

B3. (a) Explain what is meant by a *continuity condition* in the context of wave propagation, and state the conservation laws with which they are generally associated. [4]

Continuity conditions express the relationships between the wave displacements etc. either side of the interface between two regions of different material properties. [2] They generally correspond to the conservation of energy [1] and momentum [1] at the interface. *[4]*

The wave equation for the longitudinal motion within a material of density ρ and elasticity E is

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 \xi}{\partial x^2},$$

where ξ is the longitudinal displacement of the material from its rest position.

(b) Express the *dispersion relation* between the frequency ω and wavenumber k of a sinusoidal sound wave in the medium in terms of ρ and E. [2]

By inspection of the wave equation, or substitution of a trial waveform, the propagation velocity is deduced to be $v_p = \sqrt{E/\rho}$. Since $v_p = \omega/k$, we obtain

$$\omega = k\sqrt{\frac{E}{\rho}}.$$ *[2]*

The continuity conditions for the longitudinal displacement ξ of sound waves as they pass, at $x = x_0$, from a region A ($x < x_0$) of density ρ_A and elasticity E_A into another region B ($x > x_0$) of density ρ_B and elasticity E_B are

$$\begin{aligned} \xi_A(x_0, t) &= \xi_B(x_0, t), \\ E_A \frac{\partial \xi_A}{\partial x}(x_0, t) &= E_B \frac{\partial \xi_B}{\partial x}(x_0, t). \end{aligned}$$

(c) State the physical reason for each of these conditions. [2]

Continuity of ξ ensures that there are no voids or overlaps within the medium. [1] Continuity of $E\,\partial\xi/\partial x$ – i.e. of the pressure within the medium – ensures that the force upon any element falls to zero as the thickness of the element is reduced to zero, ensuring finite acceleration. [1] *[2]*

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The displacements of travelling sinusoidal sound waves in the two regions A and B may be written in the form

$$\xi_{A,B}(x,t) = a_{A,B}\cos\left(\omega t - k_{A,B}(x - x_0)\right) .$$

(d) For the situation in which a wave of angular frequency ω approaches x_0 through medium A, state the possible values for the wavenumber $k_{A,B}$ in the two regions. [2]

From the dispersion relation,

$$k = \pm\omega\sqrt{\frac{\rho}{E}}$$

where the positive and negative values correspond to the forward and backward travelling waves. In region A, both the forward-travelling incident wave and a backward-travelling reflection may be present, whereas in region B only a forward-travelling transmitted wave will occur. Hence

$$k_A = \pm\omega\sqrt{\frac{\rho_A}{E_A}}; \qquad k_B = \omega\sqrt{\frac{\rho_B}{E_B}}.$$ *[2]*

(e) Hence show that when a sound wave of amplitude a_i passes from region A to region B it results in a reflected wave of amplitude a_r where

$$\left|\frac{a_r}{a_i}\right| = \left|\frac{Z_A - Z_B}{Z_A + Z_B}\right|$$

and the acoustic impedance Z is given in terms of the density ρ and elasticity E by $Z = \sqrt{E\rho}$. [5]

If we label the incident, reflected and transmitted waves i, r and t, they will take the form [1]

$$a_i \cos[\omega t - k_A(x - x_0)]$$
$$a_r \cos[\omega t + k_A(x - x_0)]$$
$$a_t \cos[\omega t - k_B(x - x_0)]$$

where k_A is here taken to be positive.

Hence the wave displacements in the two regions will be [1]

$$\begin{aligned} \xi_A(x,t) &= a_i \cos[\omega t - k_A(x - x_0)] + a_r \cos[\omega t + k_A(x - x_0)] \\ \xi_B(x,t) &= a_t \cos[\omega t - k_B(x - x_0)] \end{aligned}$$

Applying the boundary conditions at $x = x_0$, and cancelling the common sinusoidal factors, we obtain [1]

$$a_i + a_r = a_t$$

and [1]

$$E_A k_A(a_i - a_r) = E_B k_B a_t.$$

Substituting the first of these expressions into the second, we find

$$\begin{aligned} E_A k_A(a_i - a_r) &= E_B k_B(a_i + a_r) \\ \Rightarrow a_i(E_A k_A - E_B k_B) &= a_r(E_A k_A + E_B k_B) \\ \Rightarrow \frac{a_r}{a_i} &= \frac{E_A k_A - E_B k_B}{E_A k_A + E_B k_B} = \frac{Z_A - Z_B}{Z_A + Z_B} \end{aligned}$$

where $Z \equiv \sqrt{E\rho}$ and we use the previous result that $\omega = \sqrt{E/\rho}\, k$. [1] *[5]*

(f) Ultrasound imaging depends upon the reflection of high frequency sound at the interfaces between different media. Given the data in the table below, and assuming breast and tumourous tissue to be approximately homogeneous in their acoustic properties, find the fraction of the wave *intensity* that is reflected when ultrasound is normally incident upon the interface between breast tissue and a tumour within it. [2]

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material	density (kg m^{-3})	elasticity (kPa)
breast tissue	1020	25
tumour	1040	93

Substituting the values given, we obtain [1]

$$\left|\frac{a_r}{a_i}\right| = \left|\frac{\sqrt{1020 \times 25} - \sqrt{1040 \times 93}}{\sqrt{1020 \times 25} + \sqrt{1040 \times 93}}\right| = 0.32\,.$$

The fraction of the intensity reflected is equal to the square of the amplitude reflectivity, and is hence $0.32^2 = 0.10$ *. [1]* *[2]*

(g) What other considerations would determine the overall strength of the ultrasound signal obtained when imaging a tumour within breast tissue?

[3]

The strength of the ultrasound signal would be affected by absorption within the tissue, diffraction due to the finite beam width, reflection if the ultrasound beam is not at normal incidence, diffraction and refraction by non-uniformities within the individual tissue types, focussing or defocussing due to curvature of the interface, scattering by roughness at the interface, and multi-layer interference if there are further boundaries between different tissue properties. [3 marks for something sensible] *[3]*

B4. The Boeing CH-47 *Chinook* helicopter has two synchronized, counter-rotating rotors, with axes separated by 11.9 m, which rotate at 225 revolutions per minute. Each of the overlapping rotors comprises three rotor blades.

Describe, qualitatively and when possible quantitatively, the production and propagation of sound waves from the helicopter's rotors as the aircraft flies over a stationary observer, and the sound that will be heard by the observer below. You may wish to consider

(a) the origin and nature of the sound waves, [4]

(b) the pattern of waves emitted, [6]

(c) the effects of the aircraft's motion, and [5]

(d) the consequences of having a pair of rotors. [5]

You may assume that the aircraft flies at a steady speed of 80 m s^{-1} and constant height of 160 m, and take the speed of sound in air to be 330 m s^{-1}.

The helicopter's rotor blades support the aircraft by increasing the pressure of the air beneath them [1]. As the rotors rotate, so will this pattern of elevated pressure [1]. These pressure variations will propagate away from the rotor blades as sound waves. [2] *[4]*

Directly beneath a single, unobstructed, three-bladed rotor, the propagating wavefronts would be spirals: no pressure variation would in principle be measured along the rotor axis [2]. Away from the rotor axis, however, or when the sound waves are interrupted by the presence of the aircraft fuselage or interaction with the second rotor, or when there is airflow across the rotor due to forward motion of the aircraft, this symmetry will be broken, and sound is likely to be heard. [1]. The fuselage will diffract and reflect the *travelling sound waves (and in practice probably interact nonlinearly with the nearby rotor blades). [2] The sound will repeat three times every rotor revolution, and hence have a fundamental frequency of* $3 * 225/60 = 11.25$ *Hz. [2] If the effects of the two rotors are similar, there will be a strong component at twice this frequency. [1]* *[6]*

As the aircraft approaches, passes overhead and recedes, the observed sound intensity will generally increase and then fall. [2] There will also be a Doppler shift, causing the observed frequency to be modified by a factor $1/(1 - v_1 \cos\vartheta/v_0)$*, where* v_0 *is the speed of sound,* v_1 *the speed of the aircraft, and* ϑ *the angle between the aircraft's relative position and its velocity [1]. For this case, this means that the observed fundamental frequency will vary from* $11.25/(1 + 80/330) = 9.1$ *Hz to* $11.25/(1 - 80/330) = 14.9$ *Hz. [2]* *[5]*

One might suppose that interference between the sounds from the two rotors would result in a radiation pattern similar to that from Young's double slit, taking into account the finite distance from the observer [1]. For the fundamental frequency, this would allow far-field maxima at angles φ *satisfying*

$$\sin\varphi = n\frac{\lambda}{d}$$

where n *is an integer,* $d = 11.9$ *m is the rotor separation and* $\lambda = 330/11.25 = 29.3$ *m is the fundamental wavelength, hence giving maxima when* $\sin\varphi$ *is a multiple of* $29.3/11.9 = 2.5$ *and minima at half-integer multiples [1]. Since* $-1 \le \sin\varphi \le 1$*, there will hence be no interference 'fringes' at the fundamental frequency. [1] This result assumes equal contributions from the two rotors, and should be modified to account for disturbance by the fuselage and the finite separation of the rotors in comparison with the distance to the observer. [1] Interference maxima and minima will however be possible at higher frequencies, suggesting a roughly periodic variation in harmonic content as the aircraft flies overhead. [1]* *[5]*

[As this question is intended to explore the student's assimilation of the course material as a whole, a variety of answers and depths of answer is likely, and marks may be redistributed accordingly.] *[20]*

END OF PAPER