UNIVERSITY OF SOUTHAMPTON   PHYS2023W1

SEMESTER 1 EXAMINATION 2017-2018

WAVE PHYSICS

Duration: 120 MINS (2 hours)

This paper contains 9 questions.

Answers to Section A and Section B must be in separate answer books

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language dictionary is permitted ONLY IF it is a paper version of a direct 'Word to Word' translation dictionary AND it contains no notes, additions or annotations.

Section A

A1. Explain what are meant by the terms *travelling* and *standing* waves. [2]

Travelling waves are those which maintain a constant form that is simply translated through space as time evolves [1]. Standing waves maintain a spatially fixed form, that is multiplied by an evolving function of time [1]. *[2]*

Show, with a mathematical example, how travelling waves may be superposed to form a standing wave, and vice-versa. [2]

It helps to take a sinusoidal or complex exponential wave:

$$
\begin{aligned}
A\cos kx \cos \omega t &\equiv \frac{A}{2}\left[\cos(kx-\omega t)+\cos(kx+\omega t)\right] \\
B\cos(kx-\omega t) &\equiv B\left[\cos kx \cos \omega t + \sin kx \sin \omega t\right].
\end{aligned}
$$

[1 mark for each.] *[2]*

A2. What is meant by wave *interference*? [1]

Wave interference describes the addition of wave amplitudes or displacements when two or more waves reach the same point at the same time. As the amplitude may be positive or negative, the waves may interfere constructively or destructively; the intensity is therefore not the simple sum of the component intensities. *[1]*

Describe how interference occurs in a *Michelson interferometer*, and outline how and for what purpose such an instrument may be used. Illustrate your answer with a simple diagram. [3]
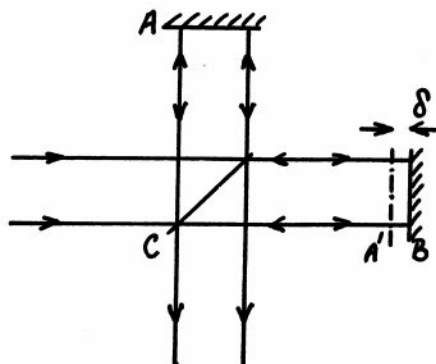
[1 mark for diagram] *[1]*

Incident light is divided by the semi-reflecting mirror C, and reflected by mirrors A and B, before being recombined at C to pass to the output. Depending upon the path difference 2δ between the two

routes, the two corresponding contributions to the output will show interference, depending upon the frequencies present. *[1]*

By recording the transmitted intensity as δ is scanned, we obtain information from which – by a Fourier transform – the spectrum of the incident light may be obtained. *[1]*

A3. Explain what is meant by the *impedance* of a medium in the context of wave propagation. [2]

The impedance is a measure of the resistance of the medium to disturbance by the process driving the wave motion. It is related to the ratio of the two properties that are conserved at an interface, and therefore determines the reflectivity at such a boundary: if the impedances are the same on both sides of the interface, the wave is not reflected. *[2]*

A wave is incident at right-angles upon the interface between two media. Show, with the aid of a formula, how the fraction of the wave power that is reflected depends upon the impedances of the two media. [2]

The fraction of the wave power that is reflected is given by

$$R = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1} \right)^2.$$

where $Z_{1,2}$ are the impedances of the two media. The fraction reflected is hence least when the impedances are similar. *[2]*

A4. The energy density of a sound wave is, like other mechanical wave motions, composed in part by the kinetic energy $\frac{1}{2}\rho(\partial\xi/\partial t)^2$ and in part by the potential energy $\frac{1}{2}E(\partial\xi/\partial x)^2$, where ξ is the longitudinal displacement at position x and time t, ρ is the density of the medium and E its modulus of elasticity.

(a) By considering a sinusoidal sound wave of definite frequency, show that these two contributions are equal. [2]

If $\xi(x,t) = \xi_0 \cos(kx - \omega t + \varphi)$, the kinetic energy density will be

$$T = \frac{1}{2}\rho\omega^2\xi_0^2 \sin^2(kx - \omega t + \varphi)$$ *[0.5]*

and the potential energy density will be

$$U = \frac{1}{2}Ek^2\xi_0^2 \sin^2(kx - \omega t + \varphi).$$ *[0.5]*

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Since $\omega/k = v_p = \sqrt{E/\rho}$, we find that

$$\rho\omega^2 = Ek^2$$

and hence the two energy contributions are equal.   [1]

(b) Given that the acoustic intensity is equal to the product of mean energy density and wave speed, find the amplitude of displacement for sound waves in sea water that correspond to the limit of dolphin hearing, $10^{-14}\,\mathrm{W\,m^{-2}}$ at a frequency of 50 kHz. The density of sea water may be taken to be 1025 kg m^{-3}, the modulus of elasticity $E \approx 2.3 \times 10^9$ Pa, and the wave speed is given by $v = \sqrt{E/\rho}$.   [2]

The wave intensity I is given by the product of the mean energy density and the wave speed – that is,

$$I = \sqrt{\frac{E}{\rho}} \times \frac{1}{2}\rho\omega^2\xi_0^2 = \sqrt{E\rho}\,\frac{\omega^2\xi_0^2}{2}.$$ [1]

This may be rearranged to give

$$\xi_0 = \sqrt{\frac{2I}{\omega^2\sqrt{E\rho}}}$$

so, with the values given,

$$\xi_0 = \sqrt{\frac{2 \times 10^{-14}\,\mathrm{W\,m^{-2}}}{(2\pi \times 5 \times 10^4\,\mathrm{s^{-1}})^2\,\sqrt{(2.3 \times 10^9\,\mathrm{Pa})(1025\,\mathrm{kg\,m^{-3}})}}} = 3.6 \times 10^{-16}\,\mathrm{m}.$$ [1]

For the sound waves addressed here, the phase velocity $v_p = \sqrt{E/\rho}$.

A5. What is meant by an *operator* in the context of wave motion?   [2]

An operator is a mathematical operation that can be applied to a wavefunction to determine observable properties of the wavefunction such as the frequency, wavenumber or bandwidth. The observable is found by applying the operator to the wavefunction according to the general operator equation

$$O\psi(x,t) = \hat{O}\psi(x,t).$$ [2]

Express the frequency and wavenumber operators $\hat{\omega}$ and $\hat{k}$ in terms of differential functions and, for one of them, show that application to a complex exponential travelling wave correctly yields the wave frequency and wavenumber.   [2]

For the sign convention that $\psi(x, t) = \psi_0 \exp[\mathrm{i}(kx - \omega t)]$ has a positive frequency and wavenumber, the operators are [0.5 for each] [1]

$$\hat{\omega} = \mathrm{i}\frac{\partial}{\partial t}$$
$$\hat{k} = -\mathrm{i}\frac{\partial}{\partial x}.$$

When applied to a complex exponential travelling wave, we find

$$O\,\psi_0 \exp[\mathrm{i}(kx - \omega t)] = \mathrm{i}\left\{-\mathrm{i}\omega\,\psi_0 \exp[\mathrm{i}(kx - \omega t)]\right\}$$

which reduces to

$$O = \omega.$$ [1]

TURN OVER

Section B

B1. (a) Making clear any assumptions, derive the equations governing the flow of heat down a uniform, thin metal bar,

$$Q(x) = -\kappa A \frac{\partial \Theta}{\partial x}$$

and $$\frac{\partial \Theta}{\partial t} = -\frac{1}{C\rho A}\frac{\partial Q}{\partial x},$$

where A is the cross-sectional area of the bar, x is the coordinate measured along its length, $\Theta(x)$ and $Q(x)$ are respectively the temperature and rate of heat flow along the bar, C is the specific heat capacity of the metal, ρ its density and κ its thermal conductivity. [6]

We assume a 1-D geometry, e.g. a bar diameter that is small in comparison with the scale of the temperature variations; and that no heat enters or leaves the edges of the bar. [1]

From the definition of the thermal conductivity κ, the magnitude of the heat Q flowing per unit cross-sectional area per unit time is the product of κ and the longitudinal temperature gradient

$$\left|\frac{Q}{A}\right| = \kappa \left|\frac{\partial \Theta}{\partial x}\right|.$$ [1]

Heat flows from a hotter to a colder body, so if Q is positive in the direction of increasing x, the negative sign is required. Inserting this, and multiplying through by A, gives the first result

$$Q(x) = -\kappa A \frac{\partial \Theta}{\partial x}.$$ [1]

From the definition of the specific heat capacity, the heat δH required to raise the temperature of a mass δm of the metal by $\delta \Theta$ is

$$\delta H = C\,\delta m\,\delta\Theta,$$ [1]

where the net heat flowing into an elemental section of length δx in time δt will be the difference between that entering and that leaving,

$$\delta H = [Q(x) - Q(x+\delta x)]\,\delta t,$$ [0.5]

and where the mass δm of the element will be

$$\delta m = A\rho\,\delta x.$$ [0.5]

Combining these expressions, dividing by $C\,\delta m\,\delta t$ and taking the limits as $\delta t \to 0$ and $\delta x \to 0$ gives the second result [1]

$$\frac{\partial \Theta}{\partial t} = \lim_{\delta t,\delta x \to 0}\frac{\delta\Theta}{\delta t} = \lim_{\delta t,\delta x \to 0}\frac{\delta H}{C\,\delta m\,\delta t}$$

$$= \lim_{\delta t,\delta x \to 0}\frac{[Q(x)-Q(x+\delta x)]}{C\,\delta m} = \lim_{\delta t,\delta x \to 0}\frac{[Q(x)-Q(x+\delta x)]}{C\,A\rho\,\delta x} = -\frac{1}{C\rho A}\frac{\partial Q}{\partial x}.$$

(b) Hence derive the *diffusion* wave equation

$$\frac{\partial \Theta}{\partial t} = \frac{\kappa}{C\rho}\frac{\partial^2 \Theta}{\partial x^2}.$$

[1]

We differentiate the first equation with respect to x to give $\partial Q/\partial x$, and substitute the result into the second equation,

$$\frac{\partial \Theta}{\partial t} = -\frac{1}{C\rho A}\frac{\partial}{\partial x}\left[-\kappa A\frac{\partial \Theta}{\partial x}\right] = \frac{\kappa}{C\rho}\frac{\partial^2 \Theta}{\partial x^2}.$$

[1]

(c) Show that the diffusion wave equation does not have sinusoidal wave solutions, but may be solved by complex exponential waves of the form

$$\Theta(x,t) = \Theta_0 \exp\left[i(kx - \omega t)\right]$$

and derive the *dispersion relation*

$$k(\omega) = \pm(1 + i)\sqrt{\frac{C\rho}{2\kappa}}\ \sqrt{\omega}.$$

[5]

Substitution of a trial sinusoidal wave $\Theta(x,t) = \Theta_0 \sin(kx - \omega t + \varphi)$ gives

$$-\omega\,\Theta_0 \cos(kx - \omega t + \varphi) = (-k^2)\frac{\kappa}{C\rho}\Theta_0 \sin(kx - \omega t + \varphi)$$

[0.5]

which may be rearranged to give

$$\tan(kx - \omega t + \varphi) = \frac{C\rho}{\kappa}\frac{\omega}{k^2}.$$

[0.5]

This can only be valid at particular combinations of x and t, so the trial form is not a solution to the diffusion wave equation.

[1]

Substitution of the trial complex exponential wave $\Theta(x,t) = \Theta_0 \exp\left[i(kx - \omega t)\right]$ gives

$$-i\omega\,\Theta_0 \exp\left[i(kx - \omega t)\right] = (-k^2)\frac{\kappa}{C\rho}\Theta_0 \exp\left[i(kx - \omega t)\right]$$

[1]

which may be rearranged to give

$$k^2 = i\frac{C\rho}{\kappa}\omega$$

[1]

which, if satisfied by ω and k, is valid for all x and t.

The dispersion relation for k in terms of ω is obtained by taking the square root of the above expression,

$$k(\omega) = \pm(1 + i)\sqrt{\frac{C\rho}{2\kappa}}\ \sqrt{\omega}.$$

[1]

TURN OVER

(d) Hence derive the real, forward-travelling solution of the form

$$\Theta(x,t) = \Theta_0 \cos(\pm k_0 x - \omega t + \varphi)\exp(\mp k_0 x),$$

and show that $k_0 = \sqrt{C\rho\omega/(2\kappa)}$. [4]

A real forward-travelling solution may be obtained by superposing solutions for $+\omega$ and $-\omega$ with the positive choice of sign of the root in the expression for k [1 mark for either of first two lines, then 1 mark per line], [3]

$$\begin{aligned}
\Theta(x,t) &= \Theta_0 \exp\left[i\left(+(1+i)\sqrt{\frac{C\rho}{2\kappa}}\sqrt{\omega}x - \omega t\right)\right] + \Theta_0 \exp\left[i\left(+(1+i)\sqrt{\frac{C\rho}{2\kappa}}\sqrt{-\omega}x + \omega t\right)\right] \\
&= \Theta_0\left\{\exp\left[i\left(+(1+i)\sqrt{\frac{C\rho}{2\kappa}}\sqrt{\omega}x - \omega t\right)\right] + \exp\left[i\left(+(i-1)\sqrt{\frac{C\rho}{2\kappa}}\sqrt{\omega}x + \omega t\right)\right]\right\} \\
&= \Theta_0 \exp\left(-\sqrt{\frac{C\rho\omega}{2\kappa}}x\right)\left\{\exp\left[i\left(+\sqrt{\frac{C\rho\omega}{2\kappa}}x - \omega t\right)\right] + \exp\left[i\left(-\sqrt{\frac{C\rho\omega}{2\kappa}}x + \omega t\right)\right]\right\} \\
&= 2\Theta_0 \exp\left(-\sqrt{\frac{C\rho\omega}{2\kappa}}x\right)\cos\left(\sqrt{\frac{C\rho\omega}{2\kappa}}x - \omega t\right).
\end{aligned}$$

Hence

$$k_0 = \sqrt{\frac{C\rho\omega}{2\kappa}}.$$ [1]

(e) The end of a brass bar is heated and cooled so that its temperature alternates with 1-second period between two fixed values. Describe how the temperature variations will differ at different distances along the bar, and, given that for brass $C\rho/\kappa = 29\,180\,\text{s m}^{-2}$, find the speed with which points of maximum temperature eventually propagate. [4]

The square wave applied at the end of the bar will undergo dispersion and attenuation as it propagates down the bar. From the expression for k_0, we see that the attenuation is greater for higher frequencies. The square wave will hence become distorted, diminished, and increasingly a sinusoidal variation at the fundamental frequency. [2]

From our expression for k_0, we may determine the phase velocity, which at the fundamental frequency will be the propagation speed of the temperature maxima,

$$v_p = \frac{\omega}{k_0} = \omega\sqrt{\frac{2\kappa}{C\rho\omega}} = \sqrt{\frac{2\kappa\omega}{C\rho}}.$$ [1]

With the values given, we find a speed

$$v_p = \sqrt{\frac{2\times 2\pi\,\text{s}^{-1}}{29\,180\,\text{s m}^{-2}}} = 21\,\text{mm s}^{-1}.$$ [1]

B2.   (a) What is meant by the *Fourier transform*? How may it be defined?   [4]

The Fourier transform allows a function of time or position to be instead represented as a function of frequency or spatial frequency - i.e. by the spectrum of sinusoidal or complex exponential components into which it may be resolved.   [2]

The component with a given frequency is obtained by multiplying the function by a sine wave (or complex exponential wave) with the same (or opposite) frequency, and integrating over the range of the function, e.g.

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$f(\omega) = \int_{-\infty}^{\infty} f(t) \cos(\omega t) dt$$

where the factor of $1/\sqrt{2\pi}$ is an arbitrary choice determined by whether the aim is to symmetrize the Fourier transform and its inverse or to normalize the intensity distribution.   [2]

The figure below shows a section of a thin, flexible string of mass per unit length ρ and subject to a tension T.
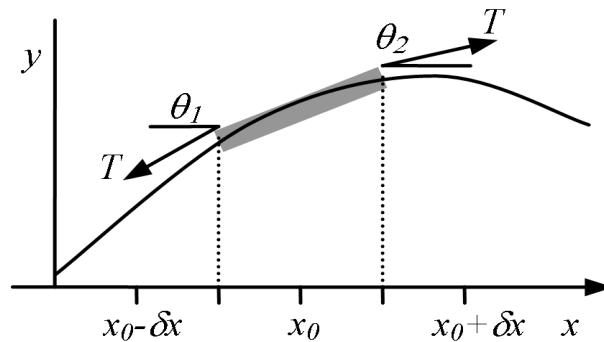
(b) By considering the net force acting on an element of the string (which, shown in grey, may be considered approximately rigid), derive the wave equation governing its transverse motion,

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho} \frac{\partial^2 y}{\partial x^2}.$$ [5]

Assuming no longitudinal motion, the mass of the element will be $\rho \delta x$.   [1]

The vertical component of tension will then be $T \sin \vartheta$, where T is assumed constant as the string is inextensible. For small ϑ this will be approximately

$$T \tan \vartheta = T \frac{\partial y}{\partial x}.$$ [1]

The net vertical component acting on the mass element will hence be

$$T \frac{\partial y}{\partial x} \Big|_{x_0 + \frac{\delta x}{2}} - T \frac{\partial y}{\partial x} \Big|_{x_0 - \frac{\delta x}{2}}$$ [1]

TURN OVER

so by Newton's second law

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho} \frac{\frac{\partial y}{\partial x}\Big|_{x_0+\frac{\delta x}{2}} - \frac{\partial y}{\partial x}\Big|_{x_0-\frac{\delta x}{2}}}{\partial x}$$

[1]

which, taking the limit $\delta x \to 0$, becomes

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho} \frac{\partial^2 y}{\partial x^2}.$$

[1]

(c) By substituting sinusoidal travelling waveform $y(x,t) = y_0 \cos(kx - \omega t + \varphi)$ into the wave equation, find the dispersion relation between k and ω, and hence show that the speed of propagation along the taut string is $\sqrt{T/\rho}$.   [3]

Substituting the form given into the wave equation, we find

$$-\omega^2 y_0 \cos(kx - \omega t + \varphi) = -k^2 \frac{T}{\rho} y_0 \cos(kx - \omega t + \varphi)$$

[1]

which reduces to

$$\omega^2 = \frac{T}{\rho} k^2$$

[0.5]

and hence

$$k = \pm \sqrt{\frac{\rho}{T}} \omega.$$

[0.5]

The phase velocity is hence

$$v_p = \frac{\omega}{k} = \sqrt{\frac{T}{\rho}}.$$

[1]

(d) Hence find the tension of a guitar string of length 520 mm and linear density 0.002 kg m^{-1} when tuned to a frequency of 220Hz (the note A$_3$).   [2]

The wavelength λ of the fundamental mode is twice the string length l. Writing the wavelength in terms of the frequency ν as $\lambda = v_p/\nu$, and using the above expression for v_p, we obtain

$$T = \rho v_p^2 = \rho(\nu\lambda)^2 = \rho(2\nu l)^2.$$

[1]

With the values given, we hence obtain

$$T = 0.002 \left[2(220\,\text{s}^{-1})(0.52\,\text{m})\right]^2 = 105\ N.$$

[1]

Ed Sheeran uses his loop station to record a rhythm track that comprises three bursts of the same guitar note A_3. The notes each last 1 s and are separated by 0.25 s, with a pause of 1.5 s before the sequence ends. The resulting 5 s track is repeated continuously throughout his song. A recording engineer uses a high resolution spectrum analyser to monitor the intensity spectrum of the loop track.

(e) Sketch the recorded signal as a function of time. [2]
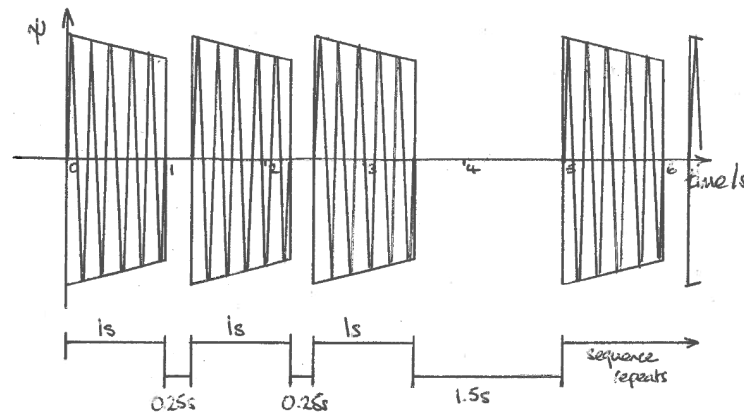
The diagram should show a series of bursts of duration 1 s, separated by 0.25 s, and identify their 5 s repetition period. It will not be possible to resolve individual 220 Hz oscillations, which can be indicated schematically. The burst envelopes need not be rectangular, the waveform within need not be a pure sinusoid, and the three bursts need not be in phase. *[2]*

(f) Hence sketch, in as much detail as possible, the spectrum that the engineer would obtain. [4]
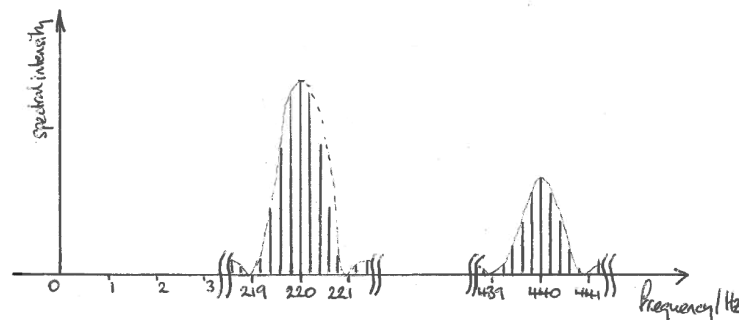
The periodic waveform will have a discrete spectrum, with components spaced by $1/(5\,\text{s}) = 0.2\,\text{Hz}$. *[1]*

TURN OVER

The 1 s duration of each note will, by the bandwidth theorem, result in a linewidth of $\sim 1\,\mathrm{Hz}$ – i.e., about 5 components. Consideration of the Fourier transform will allow more detail if the burst envelopes are rectangular or of another well-defined shape.   *[1]*

The fundamental note should be centred on 220 Hz.   *[1]*

The guitar is likely to produce overtones at multiples of 220 Hz.   *[1]*

**B3.**   (a) Explain what is meant by the *Doppler effect*.   [2]

The Doppler effect is the shift in the observed frequency of a wave when there is relative motion

between the observer and the wave source.   *[2]*

A source of waves of angular frequency ω_s moves with a velocity $\mathbf{v}$ and, at time $t = 0$, is at a position $\mathbf{r}_0$ relative to a stationary observer.

(b) Show, with the aid of a diagram, that the distance from the source to the observer at time $t \ll |\mathbf{r}_0|/|\mathbf{v}|$ will be given approximately by

$$r \approx |\mathbf{r}_0| + \frac{\mathbf{r}_0}{|\mathbf{r}_0|} \cdot \mathbf{v}t.$$ [4]

[2]
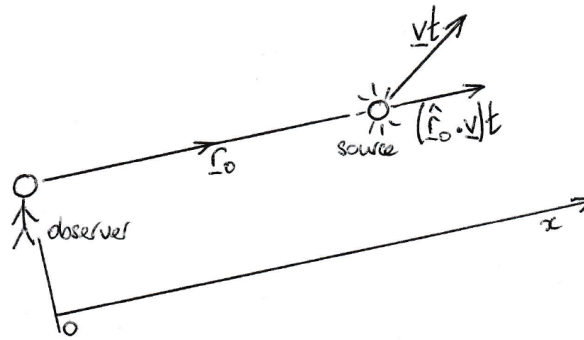
If $t \ll |\mathbf{r}_0|/|\mathbf{v}|$, the change in the bearing of the source from the observer will be negligible, and

hence the distance will change only by the component of the change of position $\mathbf{v}t$ along the unit

position vector $\mathbf{r}_0/|\mathbf{r}_0|$. Adding this correction to the initial value, we obtain the expression given.   *[2]*

(c) Show therefore that if the wave leaving the source at time t is $\psi(t)$, then that seen by the observer will be proportional to

$$\psi \left(t - t_0 - \frac{\mathbf{r}_0}{|\mathbf{r}_0|} \cdot \frac{\mathbf{v}}{c} t \right),$$

where $t_0 = |\mathbf{r}_0|/c$ and c is the speed with which the wave propagates.   [3]

We assume that the observed wave will be proportional to $\psi(t - \tau)$ [1], where $\tau = r/c$ is the time it
takes the wave to travel from the source to the observer at speed c [1]. It follows that the observed
wave will be proportional to

$$\psi \left(t - \frac{|\mathbf{r}_0|}{c} - \frac{\mathbf{r}_0}{|\mathbf{r}_0|} \cdot \frac{\mathbf{v}t}{c} \right)$$

as required [1].   *[3]*

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(d) Hence show that the observed wave will have an angular frequency $\omega_s + \delta\omega$, where

$$\frac{\delta\omega}{\omega_s} = -\frac{v_x}{c},$$

and v_x is the component of the source's velocity away from the observer.   [4]

If $\psi(t) = \psi_0 \cos \omega_s t$, then the observed wave will be proportional to

$$\psi_0 \cos\left(\omega_s\left[t - t_0 - \frac{\mathbf{r}_0}{|\mathbf{r}_0|} \cdot \frac{\mathbf{v}}{c}t\right]\right) = \psi_0 \cos\left(\omega_s\left[1 - \frac{\mathbf{r}_0}{|\mathbf{r}_0|} \cdot \frac{\mathbf{v}}{c}\right]t - \omega_s t_0\right).$$ *[1]*

The observed frequency is the coefficient of t in this expression, and hence   *[1]*

$$\omega_s + \delta\omega = \omega_s\left[1 - \frac{\mathbf{r}_0}{|\mathbf{r}_0|} \cdot \frac{\mathbf{v}}{c}\right]$$ *[1]*

where $\mathbf{v} \cdot \mathbf{r}_0/|\mathbf{r}_0| = v_x$ is the velocity component away from the observer, hence

$$\delta\omega/\omega_s = -v_x/c.$$ *[1]*

The source is an atom which, when at rest, emits or scatters photons of angular frequency ω_0. The atom emits a photon towards the observer, in whose frame the photon has an energy $\hbar\omega$. The coordinate axes may be chosen so that the x axis points from the observer to the source. Conservation of energy and momentum upon the emission of the photon leads to the relations

$$\hbar\omega = \hbar\omega_0 - m\,v_x\,\delta v$$
$$m\,\delta v = \hbar\omega/c,$$

where δv is the change in the x-component of the atom's velocity when it emits the photon, m is the mass of the atom and v_x is the mean component of its velocity away from the observer.

(e) Show that the observed angular frequency will be

$$\omega = \omega_0 \left(1 + \frac{v_x}{c}\right)^{-1}.$$ [2]

Substituting the result from conservation of momentum into the expression from conservation of energy, we find

$$\hbar\omega = \hbar\omega_0 - mv_x\hbar\frac{\omega}{c}\frac{1}{m}$$ *[1]*

so, collecting together the terms in ω,

$$\hbar\omega \left(1 + \frac{v_x}{c}\right) = \hbar\omega_0,$$ *[0.5]*

hence the required result

$$\omega = \omega_0 \left(1 + \frac{v_x}{c}\right)^{-1}.$$ *[0.5]*

(f) Show therefore that, if $v_x \ll c$, the Doppler shift of the photon due to the motion of the atom will again be

$$\delta\omega = \omega - \omega_0 \approx -\omega_0 \frac{v_x}{c}.$$ [3]

We may use the binomial expansion to obtain

$$\omega = \omega_0 \left(1 + \frac{v_x}{c}\right)^{-1} = \omega_0 \left(1 - \frac{v_x}{c} + \cdots\right)$$ *[1]*

so that, if $v_x \ll c$,

$$\omega \approx \omega_0 \left(1 - \frac{v_x}{c}\right)$$ *[1]*

and hence

$$\delta\omega \equiv \omega - \omega_0 \approx \omega_0 \left(1 - \frac{v_x}{c}\right) - \omega_0 = -\omega_0 \frac{v_x}{c}.$$ *[1]*

The Fraunhofer E line in the solar spectrum is due to absorption at wavelength $\lambda_0 = 527\,\text{nm}$ by Fe atoms in the photosphere, where the temperature T is around 5 000 K.

(g) Estimate the variation $\delta\lambda$ in the wavelength of the E line that is due to thermal motion of the Fe atoms. The r.m.s. velocity component $v_{x,rms}$ for a thermal distribution is given by $v_{x,rms}^2 = k_B T / m$, where k_B is Boltzmann's constant and the mass m of a Fe atom is 9.27×10^{-26} kg.   [2]

You may assume that $\delta\lambda/\lambda_0 = \delta\omega/\omega_0$.

Using the data given,

$$v_{x,rms} = \sqrt{\frac{k_B T}{m}} = \sqrt{\frac{1.38 \times 10^{-23} \times 5000}{9.27 \times 10^{-26}}} \,\text{m s}^{-1} = 863\,\text{m s}^{-1}.$$ *[1]*

Hence

$$\delta\lambda = \lambda_0 \frac{\delta\omega}{\omega_0} = 527 \times \frac{863}{3 \times 10^8} \,\text{nm} = 0.0015\,\text{nm}.$$ *[1]*

TURN OVER

B4. (a) Explain what is meant by *Fraunhofer diffraction.*   [3]

Diffraction is the modification of a wave by its partial obstruction by an interposed object or mask [1]. Fraunhofer diffraction is that observed sufficiently far from the mask that it may be considered a function of the observed angle alone [1]. More specifically, Fraunhofer diffraction is observed when the optical path from the source to detector depends linearly upon the position coordinates within the diffracting mask; equivalently, it is observed in what would be, in the absence of the mask, the image plane of the source [1].   [3]

(b) Show from first principles, with the aid of a diagram, that the dependence upon angle of the relative amplitude of a wave of wavelength λ, diffracted by a single slit of width a is given by the sinc function

$$a_1(\vartheta) \propto \frac{\sin\left\{\frac{\pi a}{\lambda}\sin\vartheta\right\}}{\frac{\pi a}{\lambda}\sin\vartheta}.$$ [5]
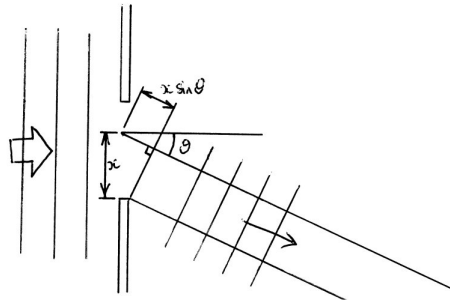
[2 marks for diagram]   [2]

If the slit, shown above with normal illumination, is taken to extend from $x = -a/2$ to $x = a/2$, then the total diffracted amplitude will be [1 mark per line]   [3]

$$
\begin{aligned}
a(\vartheta) &= a_0 \int_{-a/2}^{a/2} \exp\left(i\frac{2\pi}{\lambda}x\sin\vartheta\right)\,\mathrm{d}x \\
&= \frac{a_0}{\pi\sin\vartheta/\lambda}\frac{1}{2i}\left[\exp\left(i\frac{2\pi x\sin\vartheta}{\lambda}\right)\right]_{-a/2}^{a/2} \\
&= \frac{a_0}{\pi\sin\vartheta/\lambda}\frac{\exp\left(\frac{i\pi a\sin\vartheta}{\lambda}\right)-\exp\left(-\frac{i\pi a\sin\vartheta}{\lambda}\right)}{2i} = \frac{a_0 a}{\pi a\sin\vartheta/\lambda}\sin\frac{\pi a\sin\vartheta}{\lambda} \propto \frac{\sin\left(\frac{\pi a}{\lambda}\sin\vartheta\right)}{\frac{\pi a}{\lambda}\sin\vartheta}.
\end{aligned}
$$

(c) State the *convolution theorem* and explain how it may be used to determine the diffraction patterns of regular arrays of a basic pattern.   [3]

The convolution theorem is that the Fourier transform of the convolution of two functions is equal to the product of their individual transforms, i.e.,

$$\mathrm{FT}\{A * B\} = \mathrm{FT}\{A\} \times \mathrm{FT}\{B\}.$$ *[1]*

Because of the symmetry between a Fourier transform and its inverse, it follows that the Fourier transform of the product of two functions is equal to the convolution of their individual transforms,

$$\mathrm{FT}\{A \times B\} = \mathrm{FT}\{A\} * \mathrm{FT}\{B\}.$$ *[1]*

Since the Fraunhofer diffraction pattern is related to the Fourier transform of the mask pattern, it follows that, if the latter may be broken down into products and convolutions of functions whose Fourier transforms (i.e. diffraction patterns) are known, then the overall diffraction pattern may be written without further detailed calculation. *[1]*

The diffraction pattern of an infinite regular array of narrow slits, whose centres are separated by a distance d, is given by

$$a_2(\vartheta) = a_0 \sum_{n=0}^{\infty} \delta\left(\vartheta - n\frac{\lambda}{d}\right)$$

where ϑ is the angle through which the incident beam is diffracted.

(d) Write the transmission of a real diffraction grating, of width b and composed of narrow slits of width c spaced by a distance d, as a combination of products and convolutions of simple functions; [2]
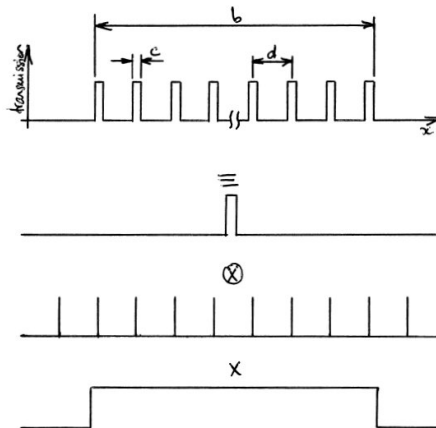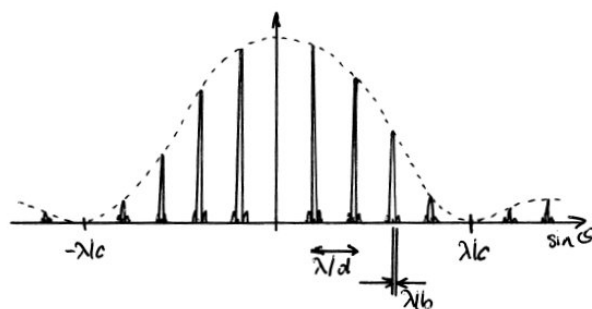

i.e., transmission of real grating $\equiv$ (single slit $*$ array of δ-functions) $\times$ wide slit. *[2]*

(e) Hence determine and sketch the diffraction pattern of such a grating. [4]

The diffraction pattern (DP) will therefore be

$$\mathrm{DP}\{\mathrm{grating}\} = \left[\mathrm{DP}\{\mathrm{single\,slit}\} \times \mathrm{DP}\{\mathrm{array\,of\,\delta\,functions}\}\right] * \mathrm{DP}\{\mathrm{wide\,slit}\}.$$ *[2]*

<div align="right">[2]</div>

(f) Given that for small angles ($\vartheta \sim \sin\vartheta$), adjacent diffraction orders are separated in angle by λ/d, and the width of each order is around λ/b, estimate the theoretical resolution of a grating spectrograph for use in first order at $\lambda = 2000$ nm when the grating parameters are $b = 100$ mm, $d = (1/300)$ mm, $c = 0.3\ \mu$m. [3]

The angular width of each diffraction order is $\sim \delta\vartheta = \lambda/b$, and the diffraction angle (i.e. angle at which the centre of the diffraction order appears) depends upon the wavelength through $\vartheta \approx \sin\vartheta = \lambda/d$. A change from ϑ to $\vartheta + \delta\vartheta$ hence corresponds to a change in wavelength of

$$\delta\lambda = \delta\vartheta\frac{\mathrm{d}\lambda}{\mathrm{d}\vartheta} = \delta\vartheta\, d = \frac{\lambda d}{b}.$$

<div align="right">[2]</div>

With the value given, we hence obtain a resolution of

$$\delta\lambda = (2000\,\mathrm{nm})\frac{1/300\,\mathrm{mm}}{100\,\mathrm{mm}} = 0.07\,\mathrm{nm}.$$

<div align="right">[1]</div>

[This assumes that the grating is optically flat across its width.]

END OF PAPER